Lecture 19

Mar 22^{nd}, 2022

Let's make the alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ sum to 2.

We'll do this by rearranging the series (in it's current arrangement, it adds to $\ln(2)$).

$$2 = \overbrace{1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}}^{(>2)} \underbrace{-\frac{1}{2}}_{}^{(<2)} + \frac{1}{17}+\frac{1}{19}+\ldots+\overset{(>2)}{\frac{1}{2k+1}} \overset{(<2)}{-\frac{1}{4}}+\ldots$$

- use the positive terms (that you haven't used yet) in order to get over 2, and a negative term to get below, and repeat.

The partial sums get closer and closer to 2.

ie) every time we use an even term $\frac{1}{2n}$, you'll be within $\frac{1}{2n}$ of 2.

So the limit of the partial sums will be 2

Note: rearranging finitely many terms does not change the sum.

→ infinitely many must be rearranged

We could do this with any target sum and any conditionally convergent series.

Absolutely convergent series converge to the same sum regardless of arrangement of terms.

Ratio Test:

Suppose $\sum_{n=0}^{\infty} a_n$ is a series such that (past some point), $a_n \neq 0$.

Then if:

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$
- <1, the series converges absolutely.
- >1, the series diverges.
- $=1$, no information.

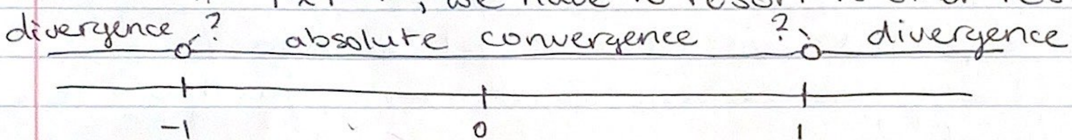
ex/ For which values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2n+3}$ converge?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{2(n+1)+3}}{\frac{x^n}{2n+3}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2n+5}\cdot\frac{2n+3}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|x\cdot\frac{2n+3}{2n+5}\right| = |x|\cdot\lim_{n\to\infty}\frac{2n+3}{2n+5}\cdot\frac{1/n}{1/n} = |x|\lim_{n\to\infty}\frac{2+3/n}{2+5/n}$$

$$= |x|\cdot\frac{2+\cancel{0}}{\cancel{2+0}} = |x|.$$

So if $|x|<1$, the series converges absolutely.
if $|x|>1$, the series diverges
if $|x|=1$, we have to resort to other tests.



If $|x| = 1$, $x=-1$ or $x=1$.

For $x=-1$, the series is $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3} = \frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\ldots$
Use Alternating Series Test.

(1) The series alternates sign because $(-1)^n$ does and $\frac{1}{2n+3}>0$.

(2) $\left|\frac{(-1)^{n+1}}{2(n+1)+3}\right| = \frac{1}{2n+5} < \frac{1}{2n+3} = \left|\frac{(-1)^n}{2n+3}\right|$

(3) $\lim_{n\to\infty}\left|\frac{(-1)^n}{2n+3}\right| = \lim_{n\to\infty}\frac{1}{2n+3} = 0$

$\therefore$ by alternating series test, $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3}$ converges

For $x=1$, the series is $\sum_{n=0}^{\infty}\frac{1^n}{2n+3} = \sum_{n=0}^{\infty}\frac{1}{2n+3} = \frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\ldots$
which diverges by the generalized p-test since $p=1-0=1\le 1$.

Thus $\sum_{n=0}^{\infty}\frac{x^n}{2n+3}$ converges if $x\in[-1,1)$
and diverges otherwise.

Root Test:

Given a series $\sum_{n=0}^{\infty} a_n$, this time we compute

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \begin{cases} <1, \text{ series converges absolutely.} \\ =1, \text{ no information} \\ >1, \text{ series diverges} \end{cases}$$

ex/ When does $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$ converge?

Note: $0^0 = 1$. ie) $f(x) = x^0$ is continuous (if $0^0 = 1$)

$$\lim_{n\to\infty} \left|\frac{x^n}{n^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{(x^n)^{\frac{1}{n}}}{(n^n)^{1/n}}\right| = \lim_{n\to\infty} \left|\frac{x^{n/n}}{n^{n/n}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{n}\right| = \lim_{n\to\infty} \frac{|x|}{n} = 0 < 1 \text{ for all } x.$$

so the series converges for all x.

ex/ for which x does the series $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$ converge?

Note: $n! = n(n-1)(n-2)\ldots(3)(2)(1)$ for $n \geq 1$ (and $0! = 1$)

Stirling's Formula: $n!$ get more/less proportiantely $\left(\frac{n}{e}\right)^n \sqrt{n}$ when n is large. (not needed for this course)

Ratio Test: (because not everything is to the n^{th} power)

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1} x^{n+1}}{(n+1)!}}{\frac{3^n x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n} \right| \qquad * (n+1)! = (n+1)n!$$

$$= \lim_{n\to\infty} \left|\frac{3x}{n+1}\right| = 3 \cdot \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \text{ for all } x$$

$\therefore$ the series converges absolutely for all x

(in fact, $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} = e^{3x}$)

ex/ When does $\sum_{n=0}^{\infty} \frac{x^{2n+2}(-1)^{3n}}{(2n)!}$ converge?

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)+2}(-1)^{3(n+1)}}{(2(n+1))!}}{\frac{x^{2n+2}(-1)^{3n}}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+4}(-1)^{3n+3}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n+2}\cdot(-1)^{3n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2(-1)^3}{(2n+2)(2n+1)}\right| = x^2\cdot\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1$$

$\therefore$ the series converges for all x by the ratio test.

ex/ When does $\sum_{n=0}^{\infty}\frac{3^n x^{n+3}}{4^n} = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n x^{n+3}$ converge?

Root Test: (a bit easier to use if we rewrite the series)

$$\sum_{n=0}^{\infty}\frac{3^n x^{n+3}}{4^n} \to \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\frac{3^n x^{n+3}}{4^n}\right|^{1/n} = \lim_{n\to\infty}\left|\frac{3x}{4}\right|\cdot\left|x^2\right|^{1/n \to 0}$$

$$= \frac{3|x|}{4}$$

or $\sum_{n=0}^{\infty}\frac{3^n x^{n+3}}{4^n} = x^3\sum_{n=0}^{\infty}\frac{3^n x^n}{4^n} \to \lim_{n\to\infty}\left|\frac{3^n x^n}{4^n}\right|^{1/n} = \lim_{n\to\infty}\left|\frac{3x}{4}\right| = \frac{3|x|}{4}$

(slightly easier)

So by the root test, the series converges when $\frac{3|x|}{4} < 1 \Rightarrow |x| < \frac{4}{3}$ and diverges when $\frac{3|x|}{4} > 1 \Rightarrow |x| > \frac{4}{3}$. What happens when $|x| = \frac{4}{3}$ (ie. $x = -\frac{4}{3}, \frac{4}{3}$)?

For $x = \frac{4}{3}$, the series is $\sum_{n=0}^{\infty}\frac{3^n\left(\frac{4}{3}\right)^{n+3}}{4^n} = \sum_{n=0}^{\infty}\frac{3^n}{4^n}\cdot\frac{4^n}{3^n}\cdot\frac{4^3}{3^3} = \sum_{n=0}^{\infty}\frac{64}{27}$, which diverges by the Divergence test since $\lim_{n\to\infty}\frac{64}{27} \neq 0$.

For $x = -\frac{4}{3}$, the series is $\sum_{n=0}^{\infty}\frac{3^n\left(-\frac{4}{3}\right)^{n+3}}{4^n} = \sum_{n=0}\frac{3^n}{4^n}\cdot(-1)^{n+3}\cdot\frac{4^n}{3^n}\cdot\frac{4^3}{3^3}$ $= \sum_{n=0}^{\infty}(-1)^{n+3}\cdot\frac{64}{27}$ which diverges by the Divergence Test since $\lim_{n\to\infty}(-1)^{n+3}\cdot\frac{64}{27} \neq 0$.

Thus the series converges when $x \in \left(-\frac{4}{3}, \frac{4}{3}\right)$.