Lecture 16 Mar. 11th, 2022

Recall: Integral Test
If $f(x)$ is decreasing on $[c,\infty)$ and integrable, then if $a_n = f(n)$ (for $n \geq c$), we have $\sum_{n=c}^{\infty} a_n$ converges exactly when the improper integral $\int_c^{\infty} f(x)dx$ converges.

• Note: $f(x)$ must be a positive function (ie. $f(x) \geq 0$)

ex/ Does $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ converge?

$$f(x) = \frac{x^2}{2^x} \geq 0 \text{ for } x \geq 0$$

$\Rightarrow f(x)$ is >0 for $x > 0$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2}{2^n} \quad \text{since } \frac{0^2}{2^0} = 0$$

→ this satisfies condition that $f(x)$ is positive

Is the function decreasing?

$$f'(x) = \frac{d}{dx}\left(\frac{x^2}{2^x}\right) = \frac{d}{dx}\left(x^2 \cdot 2^{-x}\right)$$

Quotient Rule

$$f'(x) = \frac{\frac{d}{dx}(x^2) \cdot 2^x - x^2 \cdot \frac{d}{dx}(2^x)}{(2^x)^2}$$

$$= \frac{2x \cdot \cancel{2^x} - x^2 \cdot \cancel{2^x} \ln(2)}{(2^x)^{\cancel{2}}}$$

$$= \frac{2x - x^2 \ln(2)}{2^x}$$

$$= \frac{\overset{+}{x}(2 - x\ln(2))}{\underset{+}{2^x}}$$

So $f'(x) < 0$ exactly when
$2 - x\ln(2) < 0$
$\Leftrightarrow 2 < x\ln(2)$
$\Leftrightarrow x > \frac{2}{\ln(2)} \approx 2.885\ldots$

Thus $f'(x) < 0$ and so $f(x)$ is decreasing when $x \geq 3$

Thus $\sum_{n=0}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1}{2} + \frac{2^2}{2^2} + \sum_{n=3}^{\infty} \frac{n^2}{2^n}$.

FIVE STAR.

↓

ex/ cont'd

So the original series converges exactly when $\sum_{n=3}^{\infty} \frac{n^2}{2^n}$ does.

We can apply the integral test to the last series because $f(x) = \frac{x^2}{2^x}$ satisfies the hypotheses we need on $[3, \infty)$.

ie. $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ converges as $\sum_{n=3}^{\infty} \frac{n^2}{2^n}$ converges as $\int_3^{\infty} \frac{x^2}{2^x}\,dx$ does.

$$\int_3^{\infty} \frac{x^2}{2^x}\,dx = \lim_{c\to\infty} \int_3^c \frac{x^2}{2^x}\,dx = \lim_{c\to\infty} \int_3^c x^2 \cdot 2^{-x}\,dx$$

Parts

$$= \lim_{c\to\infty} \left[x^2 \cdot \frac{-2^{-x}}{\ln(2)} \Big|_3^c - \int_3^c \frac{2x(-2^{-x})}{\ln(2)}\,dx \right] \quad \boxed{u = x^2,\ u' = 2x,\ v' = 2^{-x},\ v = \frac{-2^{-x}}{\ln(2)}}$$

$$= \lim_{c\to\infty} \left[\frac{-c^2 \cdot 2^{-c}}{\ln(2)} - \frac{-3^2 \cdot 2^{-3}}{\ln(2)} + \frac{2}{\ln(2)} \int_3^c x \cdot 2^{-x}\,dx \right]$$

Part II

$$\boxed{u = x,\ u' = 1,\ v' = 2^{-x},\ v = \frac{-2^{-x}}{\ln(2)}}$$

$$= \lim_{c\to\infty} \left[\frac{-c^2 \cdot 2^{-c}}{\ln(2)} + \frac{9 \cdot 2^{-3}}{\ln(2)} + \frac{2}{\ln(2)} \left(\frac{-x \cdot 2^{-x}}{\ln(2)} \Big|_3^c - \int_3^c \frac{-2^{-x}}{\ln(2)}\,dx \right) \right]$$

$$= \lim_{c\to\infty} \left[\frac{-c^2 \cdot 2^{-c}}{\ln(2)} + \frac{9 \cdot 2^{-3}}{\ln(2)} + \frac{2}{\ln(2)} \left(\frac{-c \cdot 2^{-c}}{\ln(2)} - \frac{-3 \cdot 2^{-3}}{\ln(2)} + \frac{1}{\ln(2)} \int_3^c 2^{-x}\,dx \right) \right]$$

$$= \lim_{c\to\infty} \left[\frac{-c^2 \cdot 2^{-c}}{\ln(2)} + \frac{9 \cdot 2^{-3}}{\ln(2)} + \frac{-c \cdot 2^{-c+1}}{[\ln(2)]^2} + \frac{3 \cdot 2^{-3+1}}{[\ln(2)]^2} + \frac{2}{[\ln(2)]^2} \cdot \frac{-2^{-x}}{\ln(2)} \Big|_3^c \right]$$

$$= \lim_{c\to\infty} \left[\underbrace{\frac{-c^2 \cdot 2^{-c}}{\ln(2)}}_{\to 0} + \frac{9 \cdot 2^{-3}}{\ln(2)} + \underbrace{\frac{-c \cdot 2^{-c+1}}{[\ln(2)]^2}}_{\to 0 \text{ (in similar way)}} + \frac{3 \cdot 2^{-2}}{[\ln(2)]^2} + \underbrace{\frac{-2^{-c+1}}{[\ln(2)]^3}}_{\to 0} + \frac{2^{-3+1}}{[\ln(2)]^3} \right]$$

$$\lim_{c\to\infty} \frac{c}{2^{c-1}} \left[\frac{\infty}{\infty}\right] = \lim_{c\to\infty} \frac{1}{\ln(2) \cdot 2^{c-1}} = 0$$

$$= \frac{9 \cdot 2^{-3}}{\ln(2)} + \frac{3 \cdot 2^{-2}}{[\ln(2)]^2} + \frac{2^{-3+1}}{[\ln(2)]^3}$$

$\Rightarrow$ the integral exists, and so $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ converges.