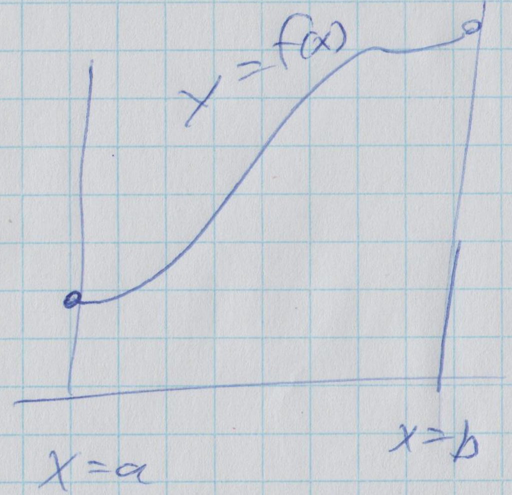
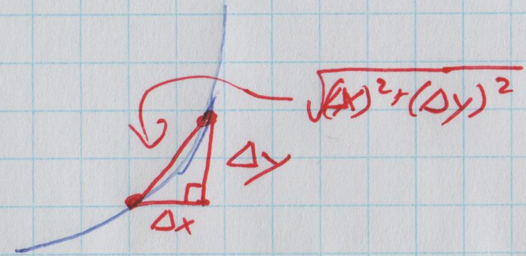
Arc-length, a.k.a. lengths of curves



How long is the graph of $y = f(x)$, $a \le x \le b$?



As we shrink Δx & Δy to being infinitesimally small,

we get a piece of arc-length, labelled

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{(dx)^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

We "add these up" using integration to get the formula

$$\begin{array}{l}\text{Arc-length of } y = f(x) \\ \text{for } a \le x \le b\end{array} = \int_a^b ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Warning: you often very hard integrals.

es $y = \tan(x)$, $0 \le x \le \frac{\pi}{4}$ $\qquad \frac{dy}{dx} = \frac{d}{dx}\tan(x) = \sec^2(x)$

$$\text{Arc-length} = \int_0^{\pi/4} \sqrt{1+\left(\frac{d}{dx}\tan(x)\right)^2}\,dx = \int_0^{\pi/4} \sqrt{1+\left(\sec^2(x)\right)^2}\,dx$$

$$= \int_0^{\pi/4} \sqrt{1+\sec^4(x)}\,dx = ??$$

Good luck!

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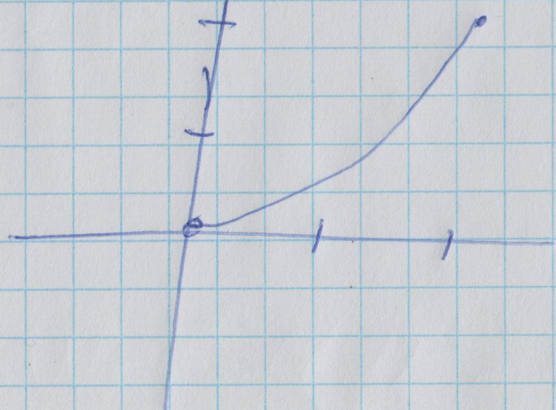
es $y = \frac{x^2}{2}$, $0 \le x \le 2$ $\qquad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{2x}{2} = x$



$$\text{Arc-length} = \int_0^2 ds = \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

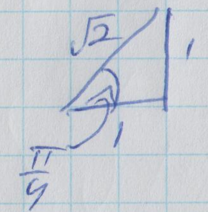
$$= \int_0^2 \sqrt{1+x^2}\,dx \qquad x = \tan(\theta), \quad dx = \sec^2(\theta)\,d\theta$$

$$= \int_{x=0}^{x=2} \sqrt{1+\tan^2(\theta)}\,d\theta = \int_{x=0}^{x=2} \sqrt{\sec^2(\theta)}\,d\theta = \int_{x=0}^{x=2} \sec(\theta)\,d\theta$$

$$= \ln(\sec(\theta)+\tan(\theta))\Big|_{x=0}^{x=2} = \ln\left(\sqrt{1+x^2}+x\right)\Big|_0^2$$

$$= \ln\left(\sqrt{1+2^2}+2\right) - \ln\left(\sqrt{1+0^2}+0\right) = \boxed{\ln\left(\sqrt{5}+2\right)} \cancel{-\ln(1)}$$

es $y = \ln(\cos(x))$, $0 \le x \le \frac{\pi}{4}$ $\qquad \cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ ③



$\Rightarrow \sec(\frac{\pi}{4}) = \frac{1}{\cos(\frac{\pi}{4})} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$

$\& \tan(\frac{\pi}{4}) = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1$

$\sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1 \quad \tan(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\cos(x))$$

$$= \frac{1}{\cos(x)} \cdot \frac{d}{dx}\cos(x)$$

$$= \frac{1}{\cos(x)}(-\sin(x)) = -\tan(x)$$

$$\text{Arc-length} = \int_0^{\pi/4} ds = \int_0^{\pi/4}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/4}\sqrt{1+(-\tan(x))^2}\,dx$$

$$= \int_0^{\pi/4}\sqrt{1+\tan^2(x)}\,dx = \int_0^{\pi/4}\sqrt{\sec^2(x)}\,dx = \int_0^{\pi/4}\sec(x)\,dx$$

$$= \ln(\sec(x)+\tan(x))\Big|_0^{\pi/4} = \ln\left(\sec(\tfrac{\pi}{4})+\tan(\tfrac{\pi}{4})\right) - \ln(\sec(0)+\tan(0))$$

$$= \boxed{\ln(\sqrt{2}+1)} - \cancel{\ln(1+0)}$$

$$\Rightarrow \quad y = \ln(x), \quad 1 \le x \le e, \quad \frac{dy}{dx} = \frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\text{Arc-length} = \int_1^e ds = \int_1^e \sqrt{1+\left(\frac{dx}{dx}\right)^2}\,dx = \int_{1}^{e} \sqrt{1+\left(\frac{1}{x}\right)^2}\,dx$$

$$= \int_1^e \sqrt{1+\frac{1}{x^2}}\,dx = \int_1^e \sqrt{\frac{x^2+1}{x^2}}\,dx = \int_1^e \frac{\sqrt{x^2+1}}{x}\,dx$$

$x = \tan(\theta)$

$dx = \sec^2(\theta)\,d\theta$

$$= \int_{x=1}^{x=e} \frac{\sqrt{\tan^2(\theta)+1}^{\;\sec(\theta)}}{\tan(\theta)} \sec^2(\theta)\,d\theta = \int_{x=1}^{x=e} \frac{\sec(\theta)}{\tan(\theta)} \sec^2(\theta)\,d\theta$$

$$= \int_{x=1}^{x=e} \frac{\sec^3(\theta)}{\tan(\theta)}\,d\theta = \int_{x=1}^{x=e} \frac{\frac{1}{\cos^3(\theta)}}{\frac{\sin(\theta)}{\cos(\theta)}}\,d\theta = \int_{x=1}^{x=e} \frac{1}{\cos^3(\theta)} \cdot \frac{\cos(\theta)}{\sin(\theta)}\,d\theta$$

$$= \int_{x=1}^{x=e} \frac{1}{\cos^2(\theta)\sin(\theta)}\,d\theta = \int_{x=1}^{x=e} \frac{\sec(\theta)}{\tan(\theta)} \left(1+\tan^2(\theta)\right) d\theta$$

$$= \int_{x=1}^{x=e} \left(\frac{\sec(\theta)}{\tan(\theta)} + \frac{\sec(\theta)}{\cancel{\tan(\theta)}} \tan^{\cancel{2}}(\theta) \right) d\theta = \int_{x=1}^{x=e} \frac{\sec(\theta)}{\tan(\theta)}\,d\theta + \int_{x=1}^{x=e} \sec(\theta)\tan(\theta)\,d\theta$$

$$= \int_{x=1}^{x=e} \frac{\frac{1}{\cos(\theta)}}{\frac{\sin(\theta)}{\cos(\theta)}}\, d\theta + \sec(\theta)\Big|_{x=1}^{x=e}$$

$$= \int_{x=1}^{x=e} \frac{1}{\cancel{\cos(\theta)}} \cdot \frac{\cancel{\cos(\theta)}}{\sin(\theta)}\, d\theta + \sqrt{1+x^2}\Big|_1^e$$

$$= \int_{x=1}^{x=e} \csc(\theta)\, d\theta + \left(\sqrt{1+e^2} - \sqrt{1+1^2}\right)$$

look it up!

$$= \ln(\csc(\theta) - \cot(\theta))\Big|_{x=1}^{x=e} + \sqrt{1+e^2} - \sqrt{2}$$

$$= \ln\left(\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right)\Big|_1^e + \sqrt{1+e^2} - \sqrt{2}$$

$\ln(\frac{a}{b}) = \ln(a) - \ln(b)$

$$= \ln\left(\frac{\sqrt{1+x^2}-1}{x}\right)\Big|_1^e + \sqrt{1+e^2} - \sqrt{2} = \ln(\sqrt{1+x^2}-1)\Big|_1^e - \ln(x)\Big|_1^e + \sqrt{1+e^2} - \sqrt{2}$$

$$= \ln(\sqrt{1+e^2}-1) - \ln(\sqrt{2}-1) - \left(\overset{1}{\ln(e)} - \overset{0}{\cancel{\ln(1)}}\right) + \sqrt{1+e^2} - \sqrt{2}$$

... so that's that ...

$x = \tan(\theta)$

$1+x^2 = 1+\tan^2(\theta) = \sec^2(\theta)$

$\Rightarrow \sec(\theta) = \sqrt{1+x^2}$

$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{x}$

$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\sqrt{1+x^2}}{x}$

$\Rightarrow \cos(\theta) = \frac{1}{\sqrt{1+x^2}}$

$\sin(\theta) = \sqrt{1-\cos^2(\theta)}$

$= \sqrt{1 - \frac{1}{1+x^2}}$

$= \sqrt{\frac{1+x^2-1}{1+x^2}}$

$= \frac{x}{\sqrt{1+x^2}}$