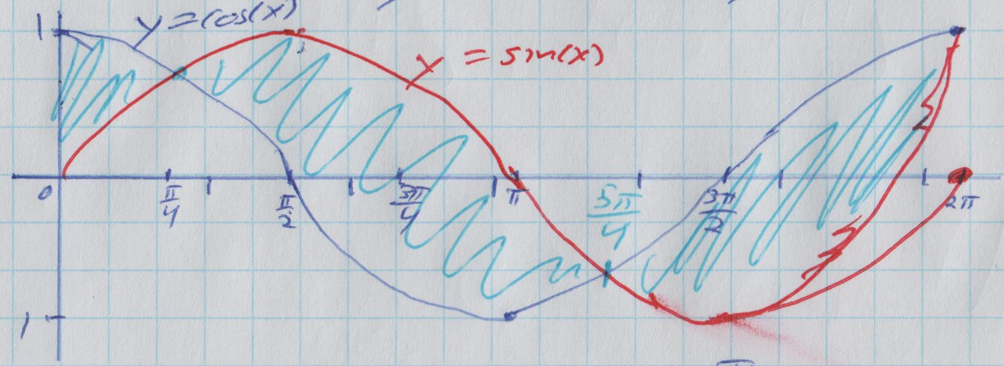
Areas and Volumes

(§9.1 & 9.3 in the text) 2022-02-16 ①

You should know that the area between two curves is given by an integral $\int_a^b (\text{upper} - \text{lower})\,dx$. May have to break this if the curves cross.

eg Area between $y = \cos(x)$ & $y = \sin(x)$, for $0 \le x \le 2\pi$.



$\sin(0) = \sin(2\pi) = 0$

$\cos(0) = \cos(2\pi) = 1$

$\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

$\sin(\frac{5\pi}{4}) = \cos(\frac{5\pi}{4}) = \frac{-1}{\sqrt{2}}$

The area is computed by

$$\int_0^{\pi/4} (\cos(x) - \sin(x))\,dx = \Big[\sin(x) + (+\cos(x))\Big]_0^{\pi/4}$$

$$+ \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\,dx \quad + \Big[-\cos(x) - \sin(x)\Big]_{\pi/4}^{5\pi/4}$$

$$+ \int_{5\pi/4}^{2\pi} (\cos(x) - \sin(x))\,dx \quad + \Big[\sin(x) + (+\cos(x))\Big]_{5\pi/4}^{2\pi}$$

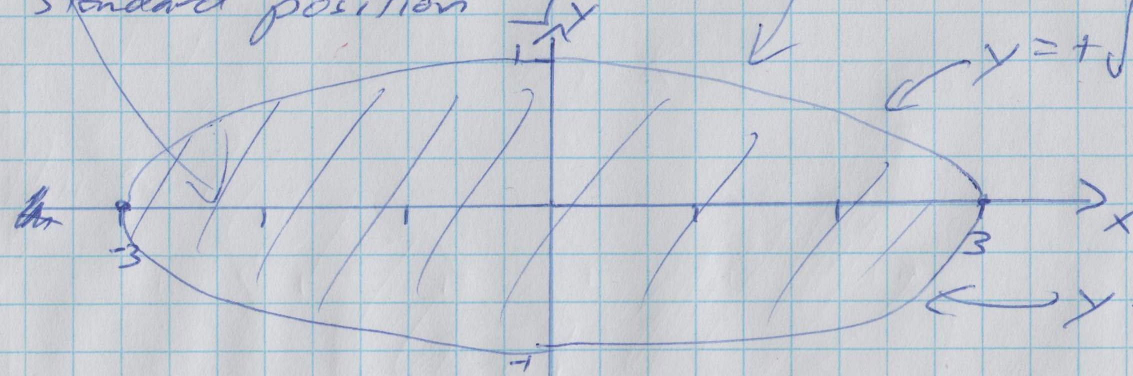
$\frac{2}{\sqrt{2}} = \sqrt{2}$

$$= \Big[\underbrace{\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right)}_{} - (0+1)\Big] + \Big[\left(-\tfrac{-1}{\sqrt{2}} - \tfrac{-1}{\sqrt{2}}\right) - \left(\tfrac{-1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\right)\Big] + \Big[(0+1) - \left(\tfrac{-1}{\sqrt{2}} + \tfrac{-1}{\sqrt{2}}\right)\Big]$$

$$= \left[\sqrt{2} - 1\right] + \left[2\sqrt{2}\right] + \left[1 + \sqrt{2}\right] = 4\sqrt{2}$$

eg Area enclosed by the ellipse $\frac{x^2}{9} + y^2 = 1$

[an ellipse with semi-major axis 3 and semiminor axis 1, in standard position]



$y = +\sqrt{1 - \frac{x^2}{9}}$

$-3 \leq x \leq 3$

$y = -\sqrt{1 - \frac{x^2}{9}}$

To realize this as the area between the graphs of two functions we have to solve for y in $\frac{x^2}{9} + y^2 = 1$

$$\Rightarrow \quad y = \pm\sqrt{1 - \frac{x^2}{9}}$$

$$\text{Area} = \int_{-3}^{3} \left[\left(+\sqrt{1 - \frac{x^2}{9}}\right) - \left(-\sqrt{1 - \frac{x^2}{9}}\right)\right] dx = \int_{-3}^{3} 2\sqrt{1 - \frac{x^2}{9}}\, dx$$

$$\left(\text{also } = 4\int_{0}^{3} \sqrt{1 - \frac{x^2}{9}}\, dx\right)$$

$$=2\int_{-3}^{3}\sqrt{1-\frac{x^2}{9}}\,dx$$

x	θ
-3	$-\frac{\pi}{2}$
3	$\frac{\pi}{2}$

$x = 3\sin(\theta)$

$dx = 3\cos(\theta)\,d\theta$

$$=2\int_{-\pi/2}^{\pi/2}\sqrt{1-\frac{3^2\sin^2(\theta)}{9}}\cdot 3\cos(\theta)\,d\theta$$

$$=2\int_{-\pi/2}^{\pi/2}\sqrt{\underbrace{1-\sin^2(\theta)}_{\cos^2(\theta)}}\cdot 3\cos(\theta)\,d\theta$$

$$=6\int_{-\pi/2}^{\pi/2}\sqrt{\cos^2(\theta)}\cdot\cos(\theta)\,d\theta$$

$$=6\int_{-\pi/2}^{\pi/2}\cos^2(\theta)\,d\theta \qquad \cos^2(\theta)=\frac{1}{2}(1+\cos(2\theta))$$

θ	u
$-\pi/2$	$-\pi$
$\pi/2$	π

$u = 2\theta$

$du = 2\,d\theta$

$\frac{1}{2}du = d\theta$

$$=6\int_{-\pi/2}^{\pi/2}\frac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$=\frac{6}{2}\int_{-\pi}^{\pi}(1+\cos(u))\cdot\frac{1}{2}\,du$$

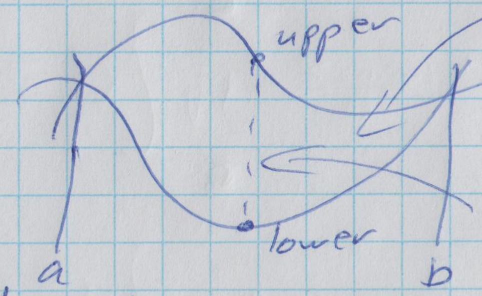
$$=\frac{6}{4}\left(u+\sin(u)\right)\Big|_{-\pi}^{\pi}$$

$$=\frac{3}{2}(\pi+0)-\frac{3}{2}(-\pi+0))$$

$$=\frac{3\pi}{2}+\frac{3\pi}{2}=3\pi$$

For area problems

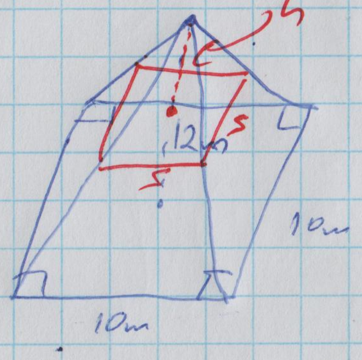
$\int_a^b (\text{upper} - \text{lower})\,dx$



we have lengths sweep out area when we integrate.

For volume problems, we have areas (of cross-sections) sweep out volume.

Let's compute the volume of a square-based pyramid where the sides at the base are 10m long and the pyramid is 12m tall.



We need nice cross-sections - we'll use horizontal because these are squares, say with sides s and a distance of h from the tip.

$\boxed{1+4} = 5$

The little sub-pyramid with height h & sides s at the base [the base is our generic cross-section] has the same proportions as the whole pyramid,

so $s:h$ as $10:12$,

<u>ie</u> $\frac{s}{h} = \frac{10}{12} \Rightarrow s = \frac{10}{12} h = \frac{5h}{6}$.

As h runs from 0 to 12 the corresponding cross-sections sweep out the volume:

Area of cross-section at h is $s^2 = \left(\frac{5h}{6}\right)^2$

$$\text{Volume} = \int_0^{12} \text{Area}\, dh = \int_0^{12} \left(\frac{5h}{6}\right)^2 dh = \int_0^{12} \frac{25}{36} h^2\, dh$$

$$= \frac{25}{36} \cdot \frac{h^3}{3} \Big|_0^{12} = \frac{25}{108} \cdot 12^3 - \cancel{\frac{25}{108} 0^2}$$

$$= \frac{25 \cdot \cancel{12^3}\,1728}{108} = \cancel{333} = \cancel{33\tfrac{1}{3}\,m^3} = 400 m^3.$$

[The general volume formula is $\frac{1}{3} h s^2 = \frac{1}{3} \cdot 12 \cdot 10^2 = 4 \cdot 10^2 = 400\, m^3$]