Integration by Parts III - More examples

eg $\int \sec^4(x)\,dx \overset{①}{=} \int \sec^2(x)\,(\underbrace{1+\tan^2(x)}_{=\sec^2(x)})\cdot dx$ $\quad w=\tan(x)$ & $dw=\sec^2(x)\,dx$

$$= \int (1+w^2)\,dw = w + \frac{w^3}{3} + C$$

$$= \tan(x) + \frac{1}{3}\tan^3(x) + C$$

② Use integration by parts: $\quad u=\sec^2(x) \quad v'=\sec^2(x)$

$v=\tan(x)$

$\boxed{\int \sec^4(x)\,dx}_{I} = \sec^2(x)\tan(x) - \int 2\sec^2(x)\tan(x)\tan(x)\,dx$

$u' = 2\sec(x)\cdot\frac{d}{dx}\sec(x)$

$= 2\sec(x)\sec(x)\tan(x)$

$= 2\sec^2(x)\tan(x)$

$= \sec^2(x)\tan(x) - 2\int \sec^2(x)\tan^2(x)\,dx$

$= \sec^2(x)\tan(x) - 2\int \sec^2(x)(\sec^2(x)-1)\,dx$

$= \sec^2(x)\tan(x) - 2\int \sec^4(x)\,dx + 2\int \sec^2(x)\,dx$

$= \sec^2(x)\tan(x) - 2\underbrace{\boxed{\int \sec^4(x)\,dx}}_{I} + 2\tan(x)$

$\therefore \; 3\boxed{\int \sec^4(x)\,dx} = \sec^2(x)\tan(x) + 2\tan(x)$

$ I$

$\therefore \; I = \int \sec^4(x)\,dx = \frac{1}{3}\sec^2(x)\tan(x) + \frac{2}{3}\tan(x) + C$

The two answers are actually equal:

second answer $\frac{1}{3}\sec^2(x)\tan(x) + \frac{2}{3}\tan(x) + C$

$= \frac{1}{3}\left(1 + \tan^2(x)\right)\tan(x) + \frac{2}{3}\tan(x) + C$

$= \frac{1}{3}\tan(x) + \frac{1}{3}\tan^3(x) + \frac{2}{3}\tan(x) + C$

$= \tan(x) + \frac{1}{3}\tan^3(x) + C$ first answer.

Moral: With trig integrals especially picking a different method may give you a different-looking answer.

es $\int_0^1 x \sinh(x)\,dx$

$u = x \qquad v' = \sinh(x)$

$u' = 1 \qquad v = \cosh(x)$

$$\boxed{\begin{aligned}\underline{\text{Recall:}}\quad \sinh(x) &= \frac{e^x - e^{-x}}{2}\\ \cosh(x) &= \frac{e^x + e^{-x}}{2}\end{aligned}}$$

$$= x\cosh(x)\Big|_0^1 - \int_0^1 1\cdot\cosh(x)\,dx$$

$$= x\cosh(x)\Big|_0^1 - \sinh(x)\Big|_0^1$$

$$= \left[x\cosh(x) - \sinh(x)\right]\Big|_0^1$$

$$= \left[1\cdot\cosh(1) - \sinh(1)\right] - \left[0\cdot\cosh(0) - \sinh(0)\right]$$

$$= \left[\frac{1}{2}\left(e+\frac{1}{e}\right) - \frac{1}{2}\left(e-\frac{1}{e}\right)\right] - \cancel{\left[0-0\right]}$$

$$= \frac{1}{2}\cdot\frac{1}{e} + \frac{1}{2}\cdot\frac{1}{e} = \frac{1}{e}$$

$$\sinh(1) = \frac{e^1 - e^{-1}}{2} = \frac{1}{2}\left(e - \frac{1}{e}\right)$$

$$\sinh(0) = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

$$\cosh(1) = \frac{e^1 + e^{-1}}{2} = \frac{1}{2}\left(e + \frac{1}{e}\right)$$

$$\cosh(0) = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

④

eg $\displaystyle\int_0^1 x\sinh(\sqrt{x})\,dx$

$u = x \quad v' = \sinh(\sqrt{x})$

$\cancel{u' = 1} \quad v = ??$

Try to simplify first... substitute

$w = \sqrt{x} \Rightarrow x = w^2 \Rightarrow dx = 2w\,dw$

Change the limits as we go along:

x	w
0	0
1	1

$$= \int_0^1 w^2\sinh(w)\,2w\,dw$$

$$= 2\int_0^1 w^3\sinh(w)\,dw$$

$u = w^3 \quad v' = \sinh(w)$

$u' = 3w^2 \quad v = \cosh(w)$

$$= 2\left[w^3\cosh(w)\Big|_0^1 - \int_0^1 3w^2\cosh(w)\,dw \right]$$

$s = w^2 \quad t' = \cosh(w)$

$s' = 2w \quad t = \sinh(w)$

$$= 2\left(1^3\cosh(1) - \cancel{0^3\cosh(0)}\right) - 6\int_0^1 w^2\cosh(w)\,dw$$

$$= 2\cdot\frac{1}{2}\left(e + \frac{1}{e}\right) - 6\left[w^2\sinh(w)\Big|_0^1 - \int_0^1 2w\sinh(w)\,dw \right]$$

$p = w \quad q' = \sinh(w)$

$p' = 1 \quad q = \cosh(w)$

$$= \left(e + \frac{1}{e}\right) - 6\left(1^2\sinh(1) - \cancel{0^2\sinh(0)}\right) + 12\int_0^1 w\sinh(w)\,dw$$

$$= \left(e + \frac{1}{e}\right) - 6\left(\frac{e - \frac{1}{e}}{2}\right) + 12\left[w\cosh(w)\Big|_0^1 - \int_0^1 1\cdot\cosh(w)\,dw \right]$$

$$= e + \frac{1}{e} - 3e + \frac{6}{e} + 12\left(1\cdot\cosh(1) - \cancel{0\cdot\cosh(0)}\right)$$

$$- 12\int_0^1 \cancel{\sinh}\cosh(w)\,dw$$

$$= -2e + \frac{7}{e} + \overset{6}{\cancel{12}}\left(\frac{e+\frac{1}{e}}{\cancel{2}}\right) - 12\sinh(w)\Big|_0^1$$

$$= -2e + \frac{7}{e} + 6e + \frac{6}{e} - 12\left(\sinh(1) - \underset{}{\overset{0}{\overset{\shortparallel}{\cancel{\sinh(0)}}}}\right)$$

$$= 4e + \frac{13}{e} - 12\left(\frac{e-\frac{1}{e}}{2}\right)$$

$$= 4e + \frac{13}{e} - 6e + \frac{6}{e} = -2e + \frac{19}{e}$$

Check my algebra & arithmetic! Just incase.

<u>es</u> Suppose $n \geq 2$. We want to compute

$$\int \cos^n(x)\,dx \qquad \text{Use parts:} \quad u = \cos^{n-1}(x) \qquad v' = \cos(x)$$

$$u' = (n-1)\cos^{n-2}(x)\cdot\frac{d}{dx}\cos(x) \qquad v = \sin(x)$$

$$\shortparallel$$

$$\int (\cos(x))^n\,dx \qquad = -(n-1)\cos^{n-2}(x)\sin(x)$$

$$\boxed{\underbrace{\int \cos^n(x)\,dx}_{I}} = \cos^{n-1}(x)\sin(x) - \int (-1)(n-1)\cos^{n-2}(x)\sin(x)\sin(x)\,dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)\,dx$$

$$\boxed{\begin{aligned}\cos^2(x)+\sin^2(x)&=1\\ \sin^2(x)&=1-\cos^2(x)\end{aligned}}$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\left(1-\cos^2(x)\right)dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\boxed{\underbrace{\int \cos^{\overset{n-2+2}{n}}(x)\,dx}_{I}}$$

$$\therefore \ [1+(n-1)]\,I = nI = n\int \cos^n(x)\,dx = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx$$

$$\Rightarrow \quad I = \int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

This "reduction" formula lets us knock down the power of cos(x) we have to deal with by 2. This idea could be repeated if $n-2 \geq 2$ still.

Next time: trigonometric integrals