Integration by Parts II

2022-01-17

More tips, tricks, and examples, and the hyperbolic functions too.

The *hyperbolic functions* are

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{"kosh"}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{"sinch"}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \text{"tanch"}$$

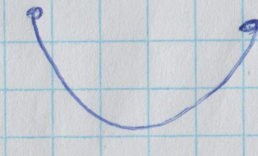
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}} \qquad \text{~~cosh~~ "seech"}$$

These have all sorts of identities eg $\cosh^2(x) - \sinh^2(x) = 1$ & interact with derivatives like (sort of) the trig functions do eg $\frac{d}{dx}\cosh(x) = \sinh(x)$ & $\frac{d}{dx}\sinh(x) = \cosh(x)$.

[Their inverses are basically natural logarithms of quadratic polynomials.]

These functions occur in hyperbolic geometry and in various other parts of math, such as solving differential equations

eg If you hang a chain between two points then the shape it hangs in is a scaled cosh curve. ("Catenary")



eg $\int x\sinh(x)\,dx$ $\quad u = x \quad v' = \sinh(x)$

$u' = 1 \quad v = \cosh(x)$

$= x\cosh(x) - \int 1\cdot\cosh(x)\,dx$

$= x\cosh(x) - \sinh(x) + C$

eg $\int x^3\cos(x^2)\,dx$ $\quad \cancel{u = x^3} \quad v' = \cos(x^2)$

$\cancel{u' = 3x^2} \quad v = ??$

Moral: try to simplify first if you can.

Simplify first using a substitution:

Try, in particular, to deal with the x^2 inside $\cos(x^2)$.

$w = x^2$ & then $dw = 2x\,dx$, so $x\,dx = \frac{1}{2}dw$.

$u = w \quad v' = \cos(w)$

$u' = 1 \quad v = \sin(w)$

$= \int x^2\cos(x^2)\cdot x\,dx = \frac{1}{2}\int w\cos(w)\,dw$

$= \frac{1}{2}\left[w\sin(w) - \int \sin(w)\,dw\right] = \frac{1}{2}\left[w\sin(w) + (+\cos(w))\right] + C$

$= \frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C$

A more precise rule of thumb for parts:

If an integrand is a product of two dissimilar functions, put whicher is of the type that occurs first on the following list into u (and the other into v'):

1. logarithmic (& inverse hyperbolic functions)
2. inverse trigonometric functions
3. polynomials (& other powers of x)
4. trigonometric functions
5. exponential & hyperbolic functions

$$\Rightarrow \quad \int e^x \sin(x)\,dx \qquad \begin{array}{ll} u = \sin(x) & v' = e^x \\ u' = \cos(x) & v = e^x \end{array}$$

$$= e^x \sin(x) - \int e^x \cos(x)\,dx \qquad \begin{array}{ll} s = \cos(x) & t' = e^x \\ s' = -\sin(x) & t = e^x \end{array}$$

$$= e^x \sin(x) - \left[e^x \cos(x) - \int (-\sin(x))\, e^x\,dx \right]$$

$$= e^x \sin(x) - e^x \cos(x) - \boxed{\int e^x \sin(x)\,dx}$$

This is what we started with... ... but with a factor of (-1) instead than 1,

ie $$\underbrace{\int e^x \sin(x)\,dx}_{I} = e^x(\sin(x) - \cos(x)) - \underbrace{\int e^x \sin(x)\,dx}_{I}$$

$$\therefore\ 2I = e^x(\sin(x) - \cos(x))$$

ie $$I = \int e^x \sin(x)\,dx = \frac{1}{2} e^x(\sin(x) - \cos(x)) + C$$

eg $$\int_1^e x\ln(x)\,dx \qquad u = \ln(x) \quad v' = x \qquad u' = \frac{1}{x} \quad v = \frac{x^2}{2}$$

$$= \frac{x^2}{2}\Big|_1^e - \int_1^e \frac{1}{x}\cdot\frac{x^2}{2}\,dx = \left(\frac{e^2}{2} - \frac{1^2}{2}\right) - \int_1^e \frac{x}{2}\,dx$$

$$= \frac{e^2-1}{2} - \frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^e = \frac{e^2-1}{2} - \frac{1}{4}(e^2-1) = \frac{e^2-1}{4}$$