Integration by Parts

... which is a technique that lets us (usually) integrate products of dissimilar functions, such as

$$\int_0^{\pi/2} x\cos(x)\,dx \,.$$

This is running the Product Rule for derivatives in reverse after a little rearranging.

Product Rule: $(f \cdot g)'(x) = \frac{d}{dx}\left[f(x)g(x)\right]$

$$= f'(x)\,g(x) + f(x)g'(x)$$

If we let $u = f(x)$ & $v = g(x)$, the Product Rule becomes $(u \cdot v)' = u' \cdot v + u \cdot v'$, which can be rearranged into $u \cdot v' = (u \cdot v)' - u' \cdot v$. Now if we integrate both sides we get

[Indefinite form of ~~the~~ Integration by Parts] $\int u \cdot v'\,dx = \int (u \cdot v)'\,dx - \int u' \cdot v\,dx$

$$= uv - \int u' \cdot v\,dx$$

[Definite form:] $\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$

Usually the key decision is to figure which part of the integrand to make the u and which to make the v'!

eg $\int_0^{\pi/2} x\cos(x)\,dx$ $u = x$ $v' = \cos(x)$

$u' = 1$ $v = \sin(x)$

Note: if we tried $u = \cos(x)$ & $v' = x$, so $u' = -\sin(x)$ & $v = \frac{x^2}{2}$, we'd be stuck integrating $\int \frac{x^2}{2}(-\sin(x))\,dx$ at the next step... which is worse than what we started with.

$$= x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\sin(x)\,dx$$

$$= \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) - 0\sin(0) - \left(-\cos(x)\right)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2}\cdot 1 - 0\cdot 0 - \left(-\cos\left(\frac{\pi}{2}\right) - (-\cos(0))\right)$$

$$= \frac{\pi}{2} - (-0 + 1) = \frac{\pi}{2} - 1 \quad \left(= \frac{\pi - 2}{2}\right)$$

Basic Rule of Thumb: Choose u and v' so that $\int u'v\,dx$ is easier (or at least no worse) than integrating $\int uv'\,dx$.

Ex $\int x^2 e^x \, dx$ $\quad u = x^2 \quad v' = e^x$

$u' = 2x \quad v = e^x$

$= x^2 e^x - \int 2x e^x \, dx$ $\quad s = 2x \quad t' = e^x$

$s' = 2 \quad t = e^x$

$= x^2 e^x - \left[2x e^x - \int 2 e^x \, dx \right]$

$= x^2 e^x - 2x e^x + 2 \int e^x \, dx$

$= x^2 e^x - 2x e^x + 2 e^x + C$

$= (x^2 - 2x + 2) e^x + C$

$\int \arctan(x) \, dx$ Problem: Is this a product ??

$= \int 1 \cdot \arctan(x) \, dx$ Answer: "Dummy" product.

$u = \arctan(x) \quad v' = 1$

$u' = \frac{1}{1+x^2} \quad v = x$

[Because we know how to take the derivative of arctan(x), but not the anti-derivative.]

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{1}{1+x^2}\cdot x\,dx$$

$$= x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

Use the substitution
$w = 1+x^2$, so
$dw = 2x\,dx$, so
$\frac{1}{2}dw = x\,dx$.

$$= x\arctan(x) - \int \frac{1}{w}\cdot\frac{1}{2}\,dw$$

$$= x\arctan(x) - \frac{1}{2}\ln(w) + C$$

$$= \boxed{x\arctan(x) - \frac{1}{2}\cdot\ln(1+x^2) + C}$$

or $= x\arctan(x) - \ln\left((1+x^2)^{1/2}\right) + C$

or $= x\arctan(x) - \ln\left(\sqrt{1+x^2}\right) + C$

Exercise: Use the dummy product to compute $\int \ln(x)\,dx$.

The dummy product trick is not useful that often:

eg $\int \sec(x)\,dx = \int 1\cdot\sec(x)\,dx$ $\qquad u=\sec(x) \quad v'=1$

$u' = \sec(x)\tan(x) \quad v = x$

$= \int x\sec(x)\tan(x)\,dx$ ... which is worse...

How do we really do this one? Not with parts, but with algebraic trickery...

$$\int \sec(x)\,dx = \int \sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

Substitute

$w = \sec(x)+\tan(x)$

$dw = (\sec(x)\tan(x)+\sec^2(x))\,dx$

$$= \int \frac{1}{w}\,dw$$

$$= \ln(w) + C$$

$$= \ln(\sec(x)+\tan(x)) + C$$

Memorize or lookup as necessary.

Next time: more tricks with parts.