The (Basic) Substitution Rule

(or, run the Chain Rule in reverse gear)

<u>Chain Rule:</u> $(f\circ g)'(x) = \frac{d}{dx} f(g(x)) = f'(g(x))\,g'(x)$

In reverse: $\int f'(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = f(g(x)) = (f\circ g)(x)$

<u>Substitution Rule:</u> $\int_a^b f(g(x))\,g'(x)\,dx$

$u = g(x)$

$du = g'(x)\,dx$

$\left[\frac{du}{dx} = g'(x)\right]$

$$= \int_{g(a)}^{g(b)} f(u)\,du$$

Alternatively $= \int_{x=a}^{x=b} f(u)\,du$ & eventually substitute back using $u = g(x)$ once you have the antiderivative of $f(u)$.

Examples: 1 $\displaystyle\int \underbrace{x\sqrt{\underbrace{1+x^2}_{u}}\;dx}_{\frac{1}{2}du}$

$u = 1+x^2$

$\frac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2}du = x\,dx$

$$= \int \sqrt{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}\,du$$ (now use Power Rule)

$$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \frac{u^{3/2}}{3} + C$$

$$= \frac{(1+x^2)^{3/2}}{3} + C$$

With indefinite integrals substitute back into the original variable: we are finding an antiderivative.

2 $$\int_0^1 x\sqrt{1+x^2}\,dx = \int_{\phi}^{2}\sqrt{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int_1^2 u^{1/2}\,du = \frac{u^{3/2}}{3}\Bigg|_1^2$$

$$= \frac{2^{3/2}}{3} - \frac{1^{3/2}}{3} = \frac{2\sqrt{2}}{3} - \frac{1}{3} = \boxed{\frac{2\sqrt{2}-1}{3}}$$

$u = 1+x^2$

$du = 2x\,dx$

$\Rightarrow x\,dx = \frac{1}{2}du$

x	u
0	1
1	2

or

$$= \frac{1}{2}\int_{x=0}^{x=1} u^{1/2}\,du = \frac{u^{3/2}}{3}\Bigg|_{x=0}^{x=1} = \frac{(1+x^2)^{3/2}}{3}\Bigg|_0^1$$

$$= \frac{2^{3/2}}{3} - \frac{1^{3/2}}{3} = \cdots = \boxed{\frac{2\sqrt{2}-1}{3}}$$

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(3) a) $\int e^{2x}\,dx = \int e^{u}\cdot\frac{1}{2}\,du = \frac{1}{2}e^{u} + C$

$u = 2x$

$du = 2\,dx$ $\qquad = \frac{1}{2}e^{2x} + C$

$\Rightarrow \frac{1}{2}\,du = dx$

b) $\int e^{2x}\,dx = \int (e^{x})^{2}\,dx = \int \underset{w}{e^{x}}\cdot\underbrace{e^{x}dx}_{dw} = \int w\,dw$

$w = e^{x}$

$dw = e^{x}dx$

$= \frac{w^2}{2} + C = \frac{(e^{x})^2}{2} + C = \frac{e^{2x}}{2} + C$

Moral: different substitution could be available

in any given substitution

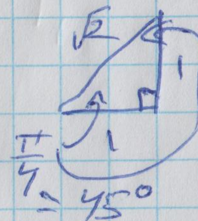
(4) $\int_0^{\pi/4} \tan(x)\,dx = \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\,dx$

$u = \cos(x)$

$du = -\sin(x)dx$

$\Rightarrow (-1)du = \sin(x)dx$

x	u
0	1
$\pi/4$	$\frac{1}{\sqrt{2}}$



$= \int_1^{1/\sqrt{2}} \frac{1}{u}\cdot(-1)\,du = \int_{1/\sqrt{2}}^{1} \frac{1}{u}\,du = \ln(u)\Big|_{1/\sqrt{2}}^{1}$

[uses $\int_a^b f(x)dx = -\int_b^a f(x)dx$] $\quad = \ln(1) - \ln(2^{-1/2}) = 0 - (-\frac{1}{2})\ln(2)$

$= \frac{1}{2}\ln(2)$

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$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{1}{u}\,(-1)\,du$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$(-1)\,du = \sin(x)\,dx$

$$= -\ln(u) + C$$

$$= \boxed{-\ln(\cos(x)) + C}$$

$$= \ln\left((\cos(x))^{-1}\right) + C$$

$$= \ln\left(\frac{1}{\cos(x)}\right) + C$$

$$= \boxed{\ln(\sec(x)) + C}$$

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$$\int_0^{\pi/2} \frac{\cos(x)\sin^2(x)}{1+\sin^2(x)}\,dx$$

$u = \cancel{\sin(x)}\ \cancel{\cos(x)}\ \sin(x)$

$du = \cancel{\cos(x)}\ \cos(x)\,dx$

$\cancel{\cos(x)\,du =}$

x	u
0	0
$\pi/2$	1

$$= \int_0^1 \frac{u^2}{1+u^2}\,du = \int_0^1 \frac{1+u^2-1}{1+u^2}\,du = \int_0^1 \left(\frac{1+u^2}{1+u^2} - \frac{1}{1+u^2}\right)du$$

$$= \int_0^1 \frac{1+u^2}{1+u^2}\,du - \int_0^1 \frac{1}{1+u^2}\,du = \int_0^1 1\,du - \int_0^1 \frac{1}{1+u^2}\,du$$

$\tan\left(\frac{\pi}{4}\right) = 1 \qquad \tan(0) = 0$

$$= u\Big|_0^1 - \arctan(u)\Big|_0^1 = (1-0) - (\arctan(1) - \arctan(0))$$

$$= 1 - \left(\frac{\pi}{4} - 0\right) = \boxed{1 - \frac{\pi}{4}}$$

<u>Next</u>: Integration by Parts