A survey of mathematical applications using Maple 10

by Gilberto E. Urroz, July 2005

This document assumes that you have read the document entitled *Getting started with Maple 10* and that you know the basic operation of the Maple 10 interface and the use of context menus for arithmetic, algebraic, and calculus operations and for graphs. It is also assumed that you know precalculus algebra, and have learned or are learning about univariate (one-variable) differential and integral calculus. We also present some examples involving multivariate (more than one variable) calculus, linear algebra, plots, data entry, differential equations, and some statistics. We will also introduce some basic operations in Maple related to manipulation of variables, data structures, etc.

Algebraic expressions

You may have noticed that most of the examples presented in the document entitled *Getting started with Maple 10* do not involve equations or assignment of values to variables. Since the intention of that document was to get the reader started in the new *point-and-click* type of mathematics (by using *context menus*), we had no need to use variable assignments. There was only one case in which an equation was used and that was to produce a two-dimensional implicit plot. In this section we will describe how to calculate expressions and apply operations to them, and how to write and use equations.

To write algebraic expressions in Maple 10 make use of the *Expression, Common Symbols,* and *Greek* palettes mainly.

<u>Calculations with *Cntl* = and context menus</u>

An expression can be "calculated" (meaning, simplified) by clicking on the *Math* input for the expression and using *Cntl* =. For example, enter the expression

$$\sqrt{\frac{x^3 - 3\,x^2y + 3\,xy^2 - y^3}{x^2 - y^2}}$$

Then, click anywhere in the expression, and type *Cntl* = . This action produces the result:

$$\sqrt{\frac{x^3 - 3\,x^2y + 3\,xy^2 - y^3}{x^2 - y^2}} = \sqrt{\frac{x^3 - 3\,x^2\,y + 3\,x\,y^2 - y^3}{x^2 - y^2}}$$

i.e., basically no simplification took place. Next, click on the expression again and do a *right-click* to produce an appropriate context menu. From the context menu select the option *Factor*. The result is now:

$$\sqrt{\frac{x^3 - 3x^2y + 3xy^2 - y^3}{x^2 - y^2}} = \sqrt{\frac{(-y+x)^2}{x+y}} = \sqrt{\frac{x^3 - 3x^2y + 3xy^2 - y^3}{x^2 - y^2}}$$

Notice that Maple retained the earlier result pushing it to the right, and showing the simplified expression in the middle of the line. You can edit this result by highlighting the last expression in the line above, together with the equal sign (=) attached to it, and pressing [*Delete*]. This, of course, produces:

$$\sqrt{\frac{x^3 - 3x^2y + 3xy^2 - y^3}{x^2 - y^2}} = \sqrt{\frac{(-y+x)^2}{x+y}}$$

This is indeed a true mathematical statement.

Now, let's try other *context menu* operations on the original expression. Click on the expression, and do a *right-click*. From the context menu select the operation *Expand,* to obtain the following result:

$$\sqrt{\frac{x^3 - 3x^2y + 3xy^2 - y^3}{x^2 - y^2}} = \sqrt{\frac{x^3}{x^2 - y^2} - \frac{3x^2y}{x^2 - y^2} + \frac{3xy^2}{x^2 - y^2} - \frac{y^3}{x^2 - y^2}}$$

In the context of the fraction contained within the root sign, *Expand* means distributing the denominator on each term in the numerator.

Let's try another *context menu* operation on the original expression (click on expression, right click). This time try *Simplify*. You will notice that this operation has the same effect as *Factor*.

Try another *context menu operation* on the original expression (click on expression, right click). This time try *Evaluate at a point*. This will produce a dialog box requesting values for the independent variables, try $x=2$ and $y=-3$, press [OK]. The Maple screen looks now as follows:

$$\sqrt{\frac{x^3 - 3\,x^2 y + 3\,x y^2 - y^3}{x^2 - y^2}} \to \sqrt{-25} = \sqrt{\frac{(-y+x)^2}{x+y}} =$$
$$\sqrt{\frac{x^3}{x^2 - y^2} - \frac{3\,x^2\,y}{x^2 - y^2} + \frac{3\,x\,y^2}{x^2 - y^2} - \frac{y^3}{x^2 - y^2}} = \sqrt{\frac{(-y+x)^2}{x+y}}$$

The general trend on using *context menus* in this case is for any new operation to displace the older ones to the right. Also, while the equal signs apply to the previous results (*Factor, Expand, Simplfy*), for the *Evaluate at a point* operation an arrow is used. However, notice that the arrow contains no information regarding the operation performed. Thus, the user himself or herself must document the results from the operations.

<u>Documenting the calculations</u>
How to accomplish this documentation of calculations:

1 – Make many copies of the expression to be operated upon, and perform one operation per copy

2 – Insert text around the operation to document was performed

To make copies of a Maple input, simply select the input expression, then do a copy-and-paste operation (*Cntl+C, Cntl+V*). For example, repeating the operations performed above, with separate outputs and documentation, one can produce the Maple worksheet shown in Figure 1.

Other operations that can be performed on the algebraic expression out of the context menu include those under the option *Constructions*. These operations are mostly for editing purposes, i.e., to produce the required expressions for documentation. Some of these expressions are shown in Figure 2.

In order to put together a *Constructions* operation with its corresponding result, we need to produce the output of the *Construction* operation, and then transform it into Maple input format by using *Cntl-drag*, i.e., selecting the output and dragging it to a position below in the worksheet while holding down the *Cntl* key. An example is illustrated in Figure 3. In this case we take the case of the derivative with respect to x, and produce an expression showing the calculation of such derivative.

Operations on an algebraic expression

We will perform a number of context menu operations for the following expression:

$$\sqrt{\frac{x^3 - 3\,x^2 y + 3\,x y^2 - y^3}{x^2 - y^2}}$$

Using *Cntl =*:

$$\sqrt{\frac{x^3 - 3\,x^2 y + 3\,x y^2 - y^3}{x^2 - y^2}} = \sqrt{\frac{x^3 - 3\,x^2\,y + 3\,x\,y^2 - y^3}{x^2 - y^2}}$$

Using *Factor* :

$$\sqrt{\frac{x^3 - 3\,x^2 y + 3\,x y^2 - y^3}{x^2 - y^2}} = \sqrt{\frac{(-y + x)^2}{x + y}}$$

Using *Expand*:

$$\sqrt{\frac{x^3 - 3\,x^2 y + 3\,x y^2 - y^3}{x^2 - y^2}} = \sqrt{\frac{x^3}{x^2 - y^2} - \frac{3\,x^2\,y}{x^2 - y^2} + \frac{3\,x\,y^2}{x^2 - y^2} - \frac{y^3}{x^2 - y^2}}$$

Using *Simplify*:

$$\sqrt{\frac{x^3 - 3\,x^2 y + 3\,x y^2 - y^3}{x^2 - y^2}} = \sqrt{\frac{(-y + x)^2}{x + y}}$$

Using *Evaluate at a point* with $x = 2$ and $y = -3$ (Also using *Approximate>10* on the result):

$$\sqrt{\frac{x^3 - 3\,x^2 y + 3\,x y^2 - y^3}{x^2 - y^2}} \rightarrow \sqrt{-25} \rightarrow 5.000000000\,\mathrm{I}$$

Figure 1 – Worksheet section documenting the context menu operations described earlier.

Additional operations on an algebraic expression

Using *Constructions>Definite Integral>x* with Lower Limit = *alpha* and Upper Limit = *beta*:

$$\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}} \rightarrow \int_{\alpha}^{\beta} \sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}}\,dx$$

Using *Constructions>Derivative>x*. Use context menu on the result with *Differentiate* to complete the derivative

$$\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}} \rightarrow \frac{\partial}{\partial x}\left(\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}}\right)$$

Using *Constructions>Evaluate at > x*, and evaluating at value = *alpha*

$$\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}} \rightarrow \left.\left(\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}}\right)\right|_{x=\alpha}$$

Using *Constructions > Integral > y*

$$\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}} \rightarrow \int \sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}}\,dy$$

Using *Constructions > Limit*, and evaluating limit at *-infinity*

$$\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}} \rightarrow \lim_{x\rightarrow -\infty}\left(\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}}\right)$$

Using *Constructions > Sum > x,* with lower limit = 0 and upper limit = n

$$\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}} \rightarrow \sum_{x=0}^{n} \sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}}$$

Figure 2 – Worksheet section documenting the *Construction* context menu operations on a given expression.

Creating an expression for the derivative

First, we use *Constructions > Derivative > x*

$$\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}} \rightarrow \frac{\partial}{\partial x}\left(\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}}\right)$$

Next, we use *Differentiate > x*, followed by *Rationalize* and *Simplify* on the subsequent outputs

Cntl-drag

$$\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}} \rightarrow$$

$$\frac{1}{2}\,\frac{\dfrac{3x^2-6xy+3y^2}{x^2-y^2}-\dfrac{2\,(x^3-3x^2y+3xy^2-y^3)\,x}{(x^2-y^2)^2}}{\sqrt{\dfrac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}}} =$$

$$\frac{(x+3y)\sqrt{\dfrac{x^2}{x+y}-\dfrac{2xy}{x+y}+\dfrac{y^2}{x+y}}}{2x^2-2y^2} = \frac{1}{2}\,\frac{(x+3y)\sqrt{\dfrac{(-y+x)^2}{x+y}}}{x^2-y^2}$$

Cntl-drag

Finally, we bring the first and third outputs together by using *Cntl-drag* and adding an equal sign (=) in between them:

$$\frac{\partial}{\partial x}\left(\sqrt{\frac{x^3-3x^2y+3xy^2-y^3}{x^2-y^2}}\right) = \frac{1}{2}\,\frac{(x+3y)\sqrt{\dfrac{(-y+x)^2}{x+y}}}{x^2-y^2}$$

Figure 3 – Details on how to create an operation out of context menu outputs.

Graphics from context menus

Details on how to produce graphics out of algebraic expressions were presented in the document entitled *Getting Started with Maple 10*. Examples shown in the following figure correspond to a surface graph (3-D plot), an implicit plot (similar to a contour plot), and a density plot of the expression

$$\sin\left(\frac{x\,y}{5}\right)$$

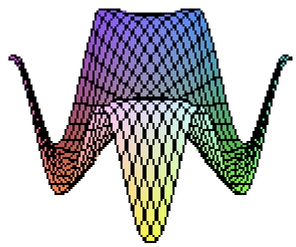
To produce the plots, click on the expression, obtain a context menu (*right-click*), and perform the operations suggested in the Figure.

Plots obtained from an algebraic expression

The particular algebraic expression under consideration involves 2 variables (*x,y*), therefore a few graphs are possible:

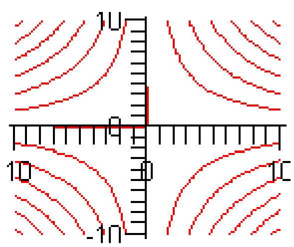
Use *Plots>3-D Plot> x,y*

$\sin\left(\frac{x\,y}{5}\right) \rightarrow$



Use *Plots>3-D Implicit Plot> x,y*

$\sin\left(\frac{x\,y}{5}\right) \rightarrow$



Use *Plots>Plot Builder*, selecting *Density plot*

$\sin\left(\frac{x\,y}{5}\right) \rightarrow$

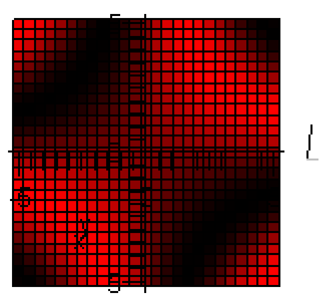


Figure 4 – Graphs out of an expression *f(x,y)* using context menus.

Working with equations
An equation, within the Maple environment, consists of two algebraic expressions joined by an equal sign. For example,

$$x^3 + 2\,x^2 = x - 2$$

Following, we show some of the context menu operations possible on this equation:

Use *Left Hand Side*
$x^3 + 2\,x^2 = x - 2 \rightarrow x^3 + 2\,x^2$

Use *Right Hand Side*
$x^3 + 2\,x^2 = x - 2 \rightarrow x - 2$

Use *Move To Left*
$x^3 + 2\,x^2 = x - 2 \rightarrow x^3 + 2\,x^2 - x + 2 = 0$

Use *Move To Right*
$x^3 + 2\,x^2 = x - 2 \rightarrow 0 = x - 2 - x^3 - 2\,x^2$

We can also use *Solve* and *Solve Numerically* to obtain the zeros of this polynomial equation:

Use *Solve*

$$x^3 + 2\,x^2 = x - 2 \rightarrow$$

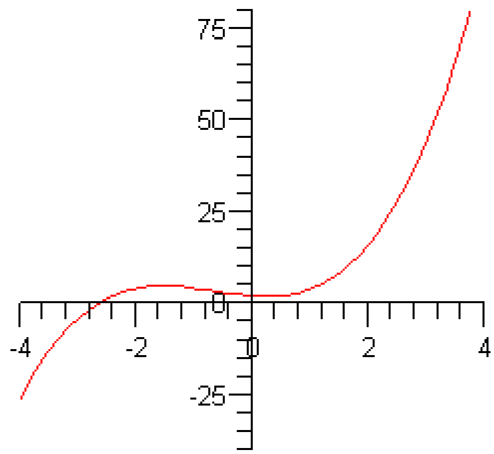
$$\left\{x = -\frac{1}{3}\left(44 + 3\sqrt{177}\right)^{(1/3)} - \frac{7}{3}\,\frac{1}{\left(44 + 3\sqrt{177}\right)^{(1/3)}} - \frac{2}{3}\right\}, \left\{x = \frac{1}{6}\left(44 + 3\sqrt{177}\right)^{(1/3)} + \frac{7}{6}\,\frac{1}{\left(44 + 3\sqrt{177}\right)^{(1/3)}} - \frac{2}{3} + \frac{1}{2}\,\mathrm{I}\sqrt{3}\left(-\frac{1}{3}\left(44 + 3\sqrt{177}\right)^{(1/3)} + \frac{7}{3}\,\frac{1}{\left(44 + 3\sqrt{177}\right)^{(1/3)}}\right)\right\}, \left\{x = \frac{1}{6}\left(44 + 3\sqrt{177}\right)^{(1/3)} + \frac{7}{6}\,\frac{1}{\left(44 + 3\sqrt{177}\right)^{(1/3)}} - \frac{2}{3} - \frac{1}{2}\,\mathrm{I}\sqrt{3}\left(-\frac{1}{3}\left(44 + 3\sqrt{177}\right)^{(1/3)} + \frac{7}{3}\,\frac{1}{\left(44 + 3\sqrt{177}\right)^{(1/3)}}\right)\right\}$$

Use *Solve Numerically*
$x^3 + 2\,x^2 = x - 2 \rightarrow \{x = -2.658967082\}$

The *Solve* operation produces all three zeros of the polynomial equation, one real, and two complex. The *Solve Numerically* operation shows the only real root, $x = -2.6589...$ A plot of the left-hand side of the equation after a *Move to the Left* operation confirms that there is only one real root (i.e., the curve crosses the x axis only once):

From the *Move To Left* result, we copy only the left-hand side, using *Cntl-drag*, then use *Plot>2-D Plot*. We use the context menu of the plot itself, to change the x range to -2 to 2, and the y range from -40 to 80:

$x^3 + 2x^2 - x + 2 \rightarrow$



Transcendental equations

These are equations involving so-called *transcendental functions* such as trigonometric, exponential, logarithms, etc. Some context menu operations with transcendental equations are shown next.

Consider the equation

$x + \tan(x) = x \cdot \sin(x) + 1$

The *Solve* operation produces only a warning message:

$x + \tan(x) = x \cdot \sin(x) + 1 \rightarrow$ `Warning, solutions may have been lost`

The *Solve Numerically* operation produces a floating-point solution:

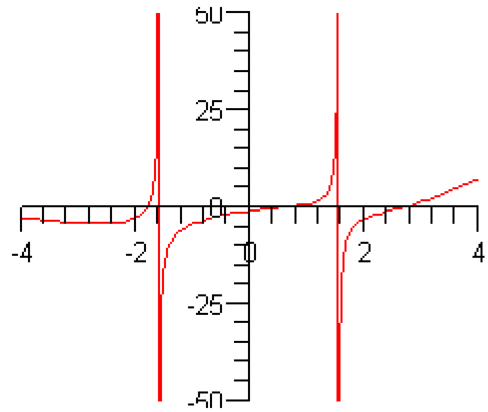
$x + \tan(x) = x \cdot \sin(x) + 1 \rightarrow \{x = 0.6383791708\}$

To see a plot, we use *Move To Left*, and then *Cntl-drag* the left-hand side of the equation. Finally, we use *Plot>2-D Plot* to produce the plot:

$x + \tan(x) = x \cdot \sin(x) + 1 \rightarrow x + \tan(x) - x\sin(x) - 1 = 0$

The plot is shown next. The scales in the x and y axes have been modified to -4 to 4, and -50 to 50, respectively. Notice that the graph shows more than one solution to this transcendental equation. The *Solve Numerically* operation produced only one of them.

$x + \tan(x) - x\sin(x) - 1 \rightarrow$



Using Maple 10 *Tutors*

Maple 10 provides a number of *Maplets* for tutoring mathematics at various levels. For example, to produce a tutor for slopes in the Precalculus collection select *Tools >Tutors >Precalculus >Slopes*). The tutor is shown below.

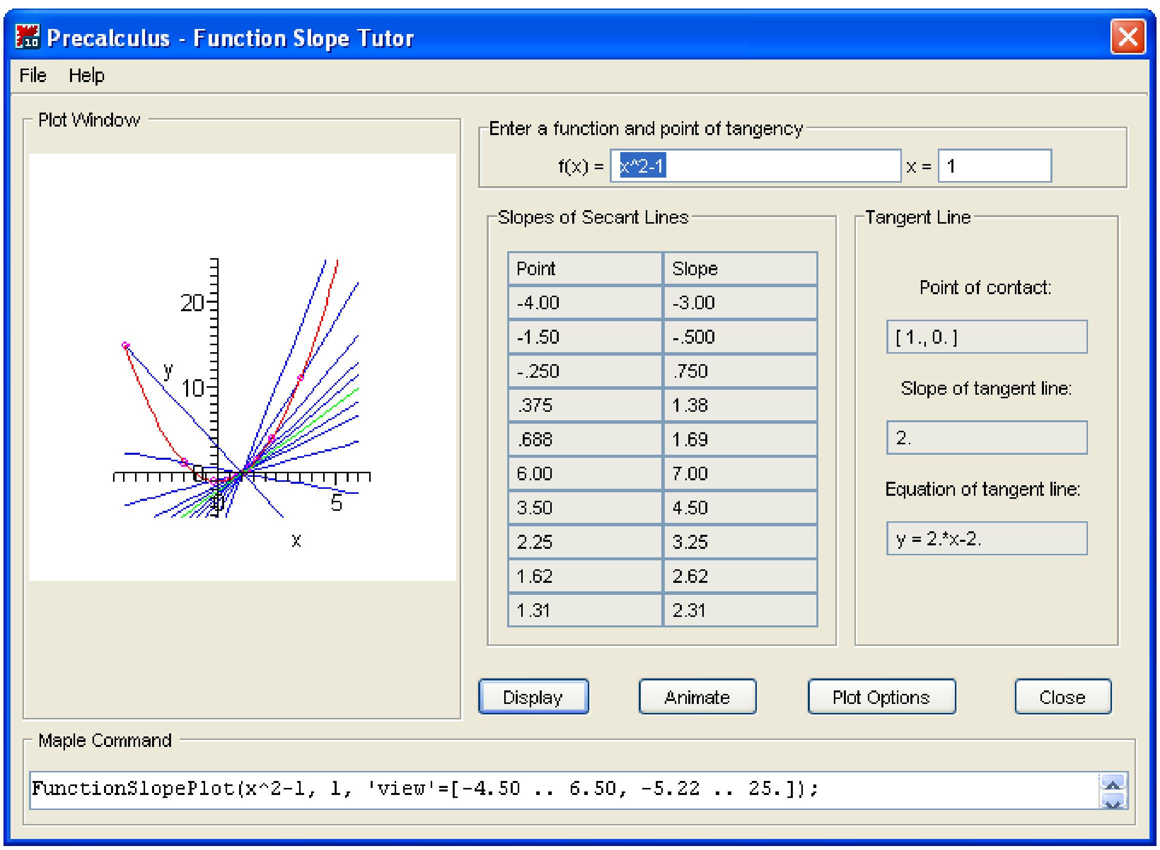


Figure 5 – Precalculus- Function Slope Tutor

The default example given corresponds to the function *f(x)* = x^2*-1*. The tutor shows the function definition, a table of values of the function, a plot of the function and several secant lines, coordinates of a contact point, and the slope and equation of the tangent line. It also has buttons to display the fixed plot, animate the drawing of secants, change the plot options, and close the tutor. At the very bottom of the tutor is the Maple command corresponding to the plot shown. Click on the [Animate] button to see an animation of the secant lines going through the contact point. The animation can be used to illustrate how a secant line approximates the tangent line at a point.

To see a different function and point of tangency, try *f(x) = sin(x)*, and use *x = -1* as the point of contact. Click on the [Animate] button to see the animation of the secant lines about the contact point.

Explore other *Tutors* of interest to you. The option *Tools>Tutors* allows access to the following subjects:

- **Precalculus**
  - Compositions...
  - Conics...
  - Slopes...
  - Limits...
  - Linear Inequalities
  - Lines...
  - Polynomials...
  - Rational Functions...
  - Standard Functions...
- **Calculus – Single Variables**
  - Antiderivatives...
  - Approximate Integrals...
  - Arc Lengths...
  - Curve Analysis...
  - Derivatives...
  - Differentiation Methods...
  - Function Average...
  - Function Inverse...
  - Integration Methods...
  - Limit Methods...
  - Mean Value Theorem...
  - Newton's Method...
  - Riemann Sums...
  - Secants...
  - Surface of Revolution...
  - Tangents...
  - Taylor Approximation...
  - Volumes of Revolution...
- **Calculus – Multi-variable**
  - Approximate Integration...
  - Cross Sections...
  - Directional Derivatives...
  - Gradients...
  - Taylor Series...
- **Linear Algebra**
  - Eigenvalues...
  - Eigenvalue Computation...
  - Eigenvector Computation...
  - Gauss-Jordan Elimination...
  - Gaussian Elimination...
  - Linear Systems...
  - Linear Transforms...
  - Matrix Inverse...
  - Solving Linear Systems...
- Vector Calculus
  - Space Curves...
  - Vector Fields...

Using Maple 10 *Assistants*

The option *Tools* > *Assistants* provides access to a number of Maplets for a variety of purposes. These Maplets are designed to provide a solution to a specific problem or process. The *Asisstants* available are:

- Curve Fitting...
- Data Analysis...
- Import Data...
- Installer Builder...
- Library Browser...
- Maplet Builder...
- Matrix Builder...
- ODE Analyzer...
- Optimization...
- Plot Builder...
- Unit Converter...

For example, the *Unit Converter* assistant is straigthforward to use as illustrated in the following figure.

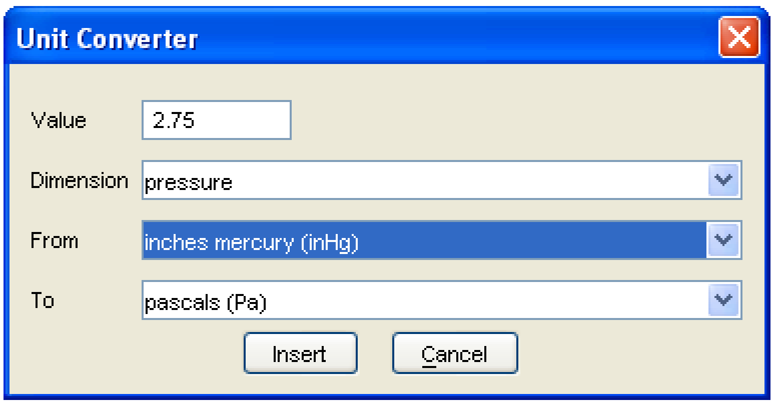


Figure 6 – An example of unit conversion using Maple's *Unit Converter* Assistant

At this point, pressing [Insert] will inset the value 9312.567434 at the current cursor location in your Maple worksheet, indicating that *2.75 inHg = 9312.567434 Pa*.

The *Import Data...* Assistant is useful for reading data from files into Maple. The *Curve Fitting* and *Data Analysis* Assistants are applicable to statistical analysis. The *Installer Builder* Assistant can be used to create Maple toolboxes (collections of functions for a specific purpose). The *Library Browser* assistant allows access to Maple's library of functions. The *Maplet Builder* helps in putting together Maplets for specific applications. The *Matrix Builder* Assistant can be used to put together matrices for linear algebra problems. The *ODE Analyzer* Assistant is helpful in solving Ordinary Differential Equations. The *Optimization* Assistant applies to problems of maximization or minimization of objective functions subject to constraints. The *Plot Builder* Assistant is useful for creating plots (many examples of using the *Plot Builder* are given in the document entitled *Getting Started with Maple 10*).

Use of the Assistant is intuitive and they're provided with a [Help] button to explain their operation. The reader is invited to explore these *Assistants* on their own.

Assigning values to variables
In an earlier section related to manipulation and solution of equations we used the equal sign (=) to build an equation in Maple. While in many computer programs the equal sign is used to assign values to variables, in Maple the *assignment symbol* is the combination **:=** (colon-equal sign). Thus, a statement such as:

Vel:=2.75

represents the assignment of the value *2.75* to the variable *Vel*.

The following is an example of variable assignments used in calculating an expression.

Assignment to variables

Example of variable assignments - Calculating cross-sectional area in a circular cross section of diameter *Diam* for an open channel flowing at a depth *y*. The variable β is an intermediate variable used in the calculation.

$Diam := 2$

$$2 \tag{8.1}$$

$y := 1.2$

$$1.2 \tag{8.2}$$

$\beta := arccos\left(1 - \frac{2 \cdot y}{Diam}\right)$

$$1.772154248 \tag{8.3}$$

$A := \frac{Diam^2}{4} \cdot (\beta - sin(\beta) \cdot cos(\beta))$

$$1.968113428 \tag{8.4}$$

Notes:
1. Press [Enter] after each assignment. This produces an output with a label (e.g., (8.1), (8.2), etc.) automatically assigned by Maple.

2. Use the *Greek* palette (see the document entitled *Getting Started with Maple 10*) to enter Greek letters such as β.

3. In this example we used the trigonometric inverse function $arccos(x) = cos^{-1}(x)$.

Besides numerical values, variable names can be assigned expressions or equations. The following example shows operations on an expression using variables.

Variables and expressions

In this example, we assign an algebraic expression to a variable and use the variable name to operate on the expression:

$exp1 := (x-5)\cdot(x+2)\cdot(x-3)$

$$(x-5)\,(x+2)\,(x-3) \qquad (9.1)$$

$exp2 := expand(exp1)$

$$x^3 - 6\,x^2 - x + 30 \qquad (9.2)$$

$sol := solve(exp2)$

$$-2, 3, 5 \qquad (9.3)$$

$x_1 := sol_1$

$$-2 \qquad (9.4)$$

$x_2 := sol_2$

$$3 \qquad (9.5)$$

$x_3 := sol_3$

$$5 \qquad (9.6)$$

$ssol := \sum_{k=1}^{3} x_k$

$$6 \qquad (9.7)$$

$S_1 := \int exp2 \, \mathrm{d}x$

$$\frac{1}{4}\,x^4 - 2\,x^3 - \frac{1}{2}\,x^2 + 30\,x \qquad (9.8)$$

$S_2 := \int_0^2 exp1 \, \mathrm{d}s$

$$2\,(x-5)\,(x+2)\,(x-3) \qquad (9.9)$$

Notes:

1. Use * for multiplication. It will show up as a dot in the input expression.
2. We used functions *expand* and *solve* that apply to algebraic expressions.
3. Variable *sol* contains the solutions to the equation *exp12 = 0, i.e.,* $x^3-6x^2-x+30=0$.
4. Since there are three solutions, we extract them separately with the statements x_1 = sol_1, etc.
5. To enter sub-indices use *Shift-underline* (*Shift* _).
6. The summation and integral symbols were obtained from the *Expression* palette

The following example shows how to use variable assignments with equations.

Variables and equations

In this example we assign an equation to a variable name and then operate on the variable name:

$eq1 := \xi \cdot (\xi - 2) \cdot (\xi + 3) = (\xi + 1) \cdot \xi$

$$\xi (\xi - 2) (\xi + 3) = (\xi + 1) \xi \quad (10.1)$$

$eq2 := expand(eq1)$

$$\xi^3 + \xi^2 - 6\xi = \xi^2 + \xi \quad (10.2)$$

$LHSEq := lhs(eq2)$

$$\xi^3 + \xi^2 - 6\xi \quad (10.3)$$

$RHSEq := rhs(eq2)$

$$\xi^2 + \xi \quad (10.4)$$

$eq3 := LHSEq - RHSEq = 0$

$$\xi^3 - 7\xi = 0 \quad (10.5)$$

$sol1 := solve(eq1)$

$$0, \sqrt{7}, -\sqrt{7} \quad (10.6)$$

$sol2 := solve(eq2, \xi)$

$$0, \sqrt{7}, -\sqrt{7} \quad (10.7)$$

$sol3 := solve(eq3, \{\xi\})$

$$\{\xi = 0\}, \{\xi = \sqrt{7}\}, \{\xi = -\sqrt{7}\} \quad (10.8)$$

$sol4 := fsolve(eq3)$

$$-2.645751311, 0., 2.645751311 \quad (10.9)$$

$sol5 := fsolve(eq1, \xi)$

$$-2.645751311, 0., 2.645751311 \quad (10.10)$$

$sol6 := fsolve(eq2, \{\xi\})$

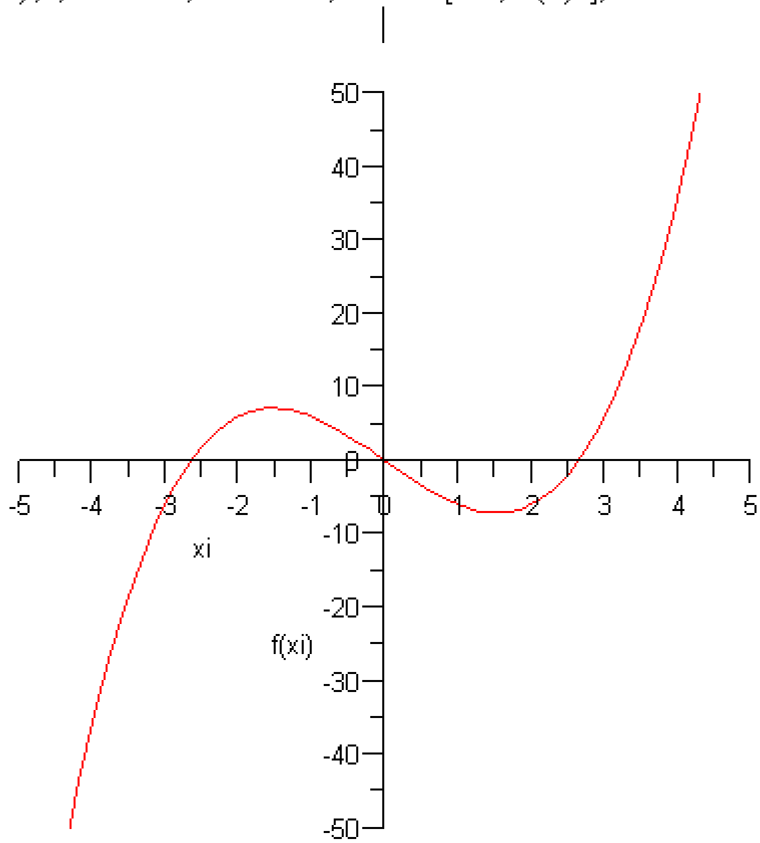
$$\{\xi = -2.645751311\}, \{\xi = 0.\}, \{\xi = 2.645751311\} \quad (10.11)$$

Notes:

1. The function *expand* applies to equations expanding each side of the equation separately.
2. Functions *lhs* and *rhs* extract the left-hand side and right-hand side of the equation, respectively.
3. We show three different versions of the *solve* command. The all provide the same solutions, but the last version shows the independent variable name in the solution.
4. We also show three different versions of the *fsolve* command. The *fsolve* command provides numerical values for the solutions (i.e., floating-point values), whereas the *solve* command shows symbolic solutions.

To illustrate the location of the solutions for the equation, we plot the left-hand side of *eq3*, in which the equation takes the form $f(\xi) = 0$, i.e., we would plot $f(\xi) = \xi^3-7\xi$. The *plot* command and the resulting graph are shown next:

plot(*lhs*(*eq3*), ξ = −5 ..5, −50 ..50, *labels* =["xi", "f(xi)"], *axes* = *normal*)



The *plot* command, in this case, includes 5 arguments:

1. *lhs(eq3)* - The function to be plotted
2. *x = -5..5* - Range of the independent variable to be shown in the plot
3. *-50..50* - Range of the dependent variable to be shown in the plot
4. *labels* = [“xi”, “f(xi)”] - The axes labels
5. *axes = normal* - Type of axes shown

Some of these arguments are optional, for example, you could omit the argument *axes=normal* since this option is the default value for axes types (other values are *boxed, frame,* and *none*). As an exercise, try the following calls to the plot command:

- *plot(lhs(eq3), ξ = -5 .. 5, -50 .. 50, labels = ["xi", "f(xi)"], axes = normal)*
- *plot(lhs(eq3), ξ = -5 .. 5)*
- *plot(lhs(eq3), ξ = -5 .. 5, labels = ["x", "y"], axes = boxed)*
- *plot(lhs(eq3), ξ = -4 .. 4, -30 .. 30, axes = framed)*

Inline help request
To request help on a given command (if the command name is known), type the question mark followed immediately by the command name. In your Maple worksheet, for example, try:

?plot

To get information on the *plot* command. A worksheet with information on the command will be shown in your Maple interface.

As an exercise, try requesting information about some of the commands presented in the examples above (for *sum* and *int*, we actually used the palette expressions):

?expand	*?rhs*
?solve	*?sum*
?fsolve	*?int*
?lhs	

Alternatively, you can use the option *Help>Maple Help* (or, *Cntl+F1*) to get a listing of all Maple commands and obtain information on any of them.

Defining functions in Maple
The preferred way to define a function in Maple is by using the arrow operator (->). The *Expression* palette includes several items related to function definitions. These are listed next (this table is also Table 4 in the document entitled *Getting Started with Maple 10*):

Table 1 - Palette expressions and Maple Input commands for function definition and evaluation

Palette expression	Maple Input command
$f(a)$	`> f(a);`
$f(a, b)$	`> f(a,b);`
$f := x \to y$	`> f := x -> y;`
$f := (x1, x2) \to y$	`> f := (x1, x2) -> y;`
$f(x) \Big\vert_{x=a}$	`> eval(f(x),x=a);`
$\begin{cases} -x & x < 0 \\ x & x > 0 \end{cases}$	`> piecewise(x<0,-x,x>0,x);`

The right-hand side of the entries in *Table 1* show the equivalent *Maple Input* command corresponding to the function palette expressions from the *Expression* palette. In *Math* entry form, these expressions would be written simply as:

f(a)
f(a,b)
f:=x->y
f:=(x1,x2)->y
eval(f(x),x=a)
piecewise(x<0,-x,x>0,x)

The following examples show applications of the previous commands in Maple:

Defining univariate functions in Maple

First, we define a univariate function using the *arrow* operator:

$h := x \rightarrow \sin(x) + \cos(x)$

$$x \rightarrow \sin(x) + \cos(x) \tag{11.1}$$

This function can be evaluated using symbolic or numeric arguments, e.g.,

$h(a+b)$

$$\sin(a+b) + \cos(a+b) \tag{11.2}$$

$h\left(\frac{Pi}{6}\right)$

$$\frac{1}{2} + \frac{1}{2}\sqrt{3} \tag{11.3}$$

$evalf((11.3))$

$$1.366025404 \tag{11.4}$$

$h(-0.76)$

$$0.0359145656 \tag{11.5}$$

$eval(h(x), x = 2)$

$$\sin(2) + \cos(2) \tag{11.6}$$

Notes:

1. In *Math* input format, when you type the arrow operator `->` it becomes →.
2. *Pi* is a representation of the constant π.
3. The evaluation (11.3) produces a symbolic result. The next line, *evalf*((11.3)), produces the floating point equivalent. In this case, we used the label (11.3) to refer to the result.

NOTE ON LABELS: To enter a label use *Cntl+l*. This produces a dialog box where the user can type the required label. It is not necessary to type the parentheses around the label.

The following examples show differentiation and integration of the function *h(x)* defined above using the full commands (i.e., *diff* or *int*), and the palette symbols:

$diff(h(x), x)$

$$\cos(x) - \sin(x) \tag{11.7}$$

$int(h(x), x)$

$$-\cos(x) + \sin(x) \tag{11.8}$$

$int(h(x), x = a..b)$

$$\cos(a) - \sin(a) - \cos(b) + \sin(b) \tag{11.9}$$

$\frac{d}{dx}\, h(x)$

$$\cos(x) - \sin(x) \tag{11.10}$$

$\int h(x)\ dx$

$$-\cos(x) + \sin(x) \tag{11.11}$$

$\int_a^b h(x)\,dx$

$$\cos(a) - \sin(a) - \cos(b) + \sin(b) \tag{11.12}$$

Composition of functions is illustrated with the following examples:

$r := x \rightarrow exp(x)$

$$x \rightarrow e^x \tag{11.13}$$

$s := x \rightarrow ln(x)$

$$x \rightarrow \ln(x) \tag{11.14}$$

$h(r(x))$

$$\sin(e^x) + \cos(e^x) \tag{11.15}$$

$r(h(xi))$

$$e^{(\sin(\xi) + \cos(\xi))} \tag{11.16}$$

$s(r(x))$

$$\ln(e^x) \tag{11.17}$$

$simplify((11.17))$

$$\ln(e^x) \tag{11.18}$$

$simplify((11.17))$ *assuming positive*

$$x \tag{11.19}$$

Note: The functions *exp* and *ln* are inverse to each other, thus, we expect that *exp(ln(x))* = *ln(exp(x))* = *x*. However, when we attempt that composition in (11.17), we simply get

$$ln(e^x)$$

An attempt to simplify this expression with the command *simplify* produces no result [see (11.18)], unless we add the particle *assuming positive* to the *simplify* command. With this addition Maple is informed that *x* is a positive quantity and the simplification is achieved. Annoter possibility is to use

simplify((11.17)) *assuming real*

The *assume* command

In any operation Maple will assume that a symbol represents the most general type of mathematical object. For example, in a function, unless told otherwise, Maple assumes that *x* could be, in the most general case, a complex number. If we want to limit the range of *x* to the real numbers we can use the command:

assume(x,real)

Any reference to x after this *assume* command is activated will show *x* followed by a tilde, i.e., *x~*. This is to remind the user that an assumption has been made about that particular variable. Here is an example:

$assume(x > 0)$

$int\left(\frac{1}{1+x}, x\right)$

$$\ln(1 + x\sim) \qquad (11.20)$$

$diff(x^2 \cdot sin(x), x)$

$$2\,x\sim \sin(x\sim) + x\sim^2 \cos(x\sim) \qquad (11.21)$$

To find out if an assumption has been made about a symbol, use the function *hasassumptions*. To find out which assumptions have been made about a symbol, use the function *getassumptions*. Examples are shown next:

$hasassumptions(x)$

$$true \qquad (11.22)$$

$getassumptions(x)$

$$\{x\sim{::}(RealRange(Open(0), \infty))\} \qquad (11.23)$$

Although the result of *getassumptions* in (11.23) is not as straightforward as *x>0*, the expression *x~::(RealRange(Open(0),))* contains basically the same information.

To find out more about the *assume* command, type:

?assume

To remove an assumption on a variable, use, for example:

x:='x'

This statement simply indicates that the variable *x* has been redefined to the symbol *x*. After having made an assumption on *x*, above, we redefine *x* and check for any assumptions as follows:

$x := 'x'$

$$x \tag{11.24}$$

$hasassumptions(x)$

$$false \tag{11.25}$$

Plots for univariate functions

The following examples show how to use the command *plot* to plot a given function:

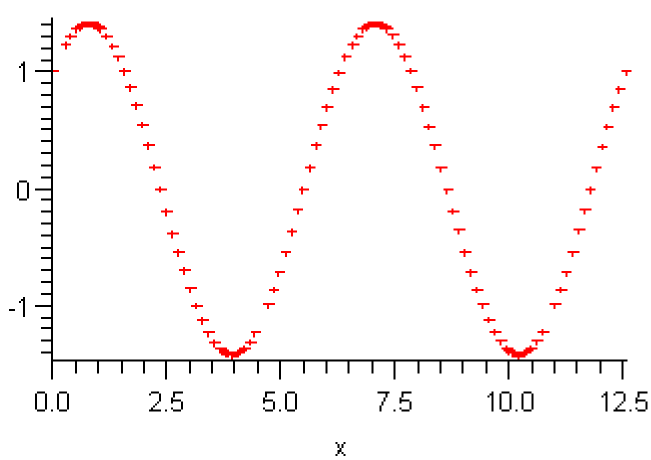
$h := x \rightarrow sin(x) + cos(x)$

$$x \rightarrow \sin(x) + \cos(x) \tag{11.26}$$

$g := x \rightarrow exp(-x) \cdot cos\left(\frac{x}{2} + \frac{Pi}{3}\right)$

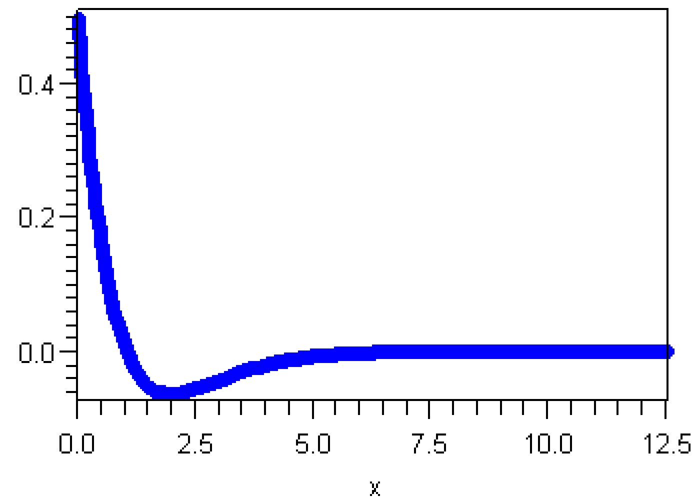
$$x \rightarrow \mathrm{e}^{(-x)} \cos\left(\frac{1}{2}x + \frac{1}{3}\pi\right) \tag{11.27}$$

$plot(h(x), x = 0..4 \cdot Pi, style = point, symbol = cross, axes = framed)$



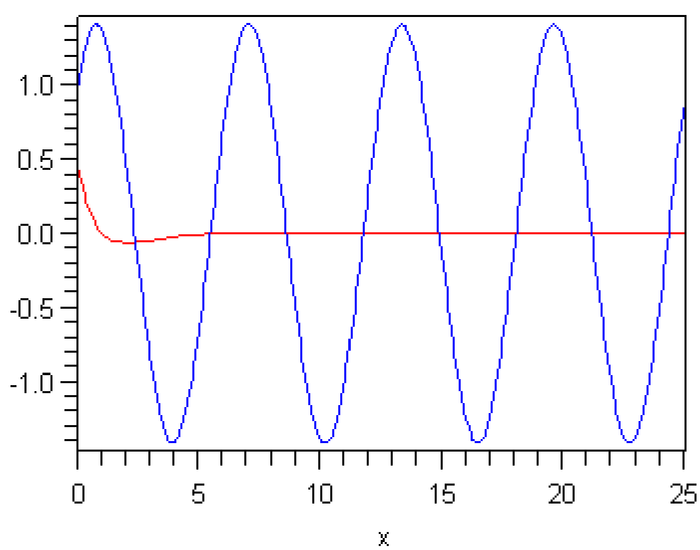
Note the arguments *style = point, symbol = cross* which control the plot style.

$plot(g(x), x = 0..4 \cdot Pi,\ style = line,\ thickness = 5,\ color = blue, axes = boxed)$



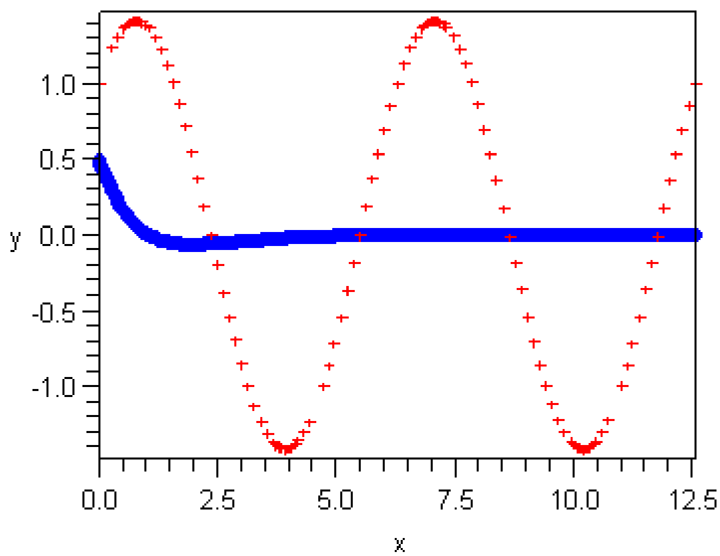
We can produce a plot of the two functions by using:

$plot(\{h(x), g(x)\}, x = 0..8 \cdot Pi, color = [red, blue], axes = boxed)$



An alternative way to produce the combined plot would be to store the two individual plots into variable names, and then use the command *plots[display]* to show the two plots together. To avoid showing the plots as we store them into variables, we'll end the corresponding commands with a colon (:). A colon at the end of a Maple command suppresses the output of the command, even though the command does get executed. The commands and resulting plot are shown next:

```
p1 := plot(h(x), x = 0..4·π, style = point, symbol = cross) :
p2 := plot(g(x), x = 0..4·π, style = line, thickness = 5, color = blue) :
plots[display]({p1, p2}, axes = boxed, labels = ["x", "y"])
```



Multivariate functions

In this section we'll show some examples for bivariate (two-variable) functions. These are extensions of the examples shown above for univariate (one-variable) functions. First, we show definitions and evaluations of bivariate functions:

restart :

$h := (x, y) \rightarrow \sin(x \cdot y)$

$$(x, y) \rightarrow \sin(x\, y) \quad (12.1)$$

$g := (x, y) \rightarrow x^2 + y^2$

$$(x, y) \rightarrow x^2 + y^2 \quad (12.2)$$

$h\left(\frac{Pi}{6}, \frac{Pi}{3}\right)$

$$\sin\left(\frac{1}{18}\pi^2\right) \quad (12.3)$$

evalf[10]((12.3),)

$$0.5212468738 \quad (12.4)$$

$g\left(\frac{Pi}{6}, \frac{Pi}{3}\right)$

$$\frac{5}{36}\pi^2 \quad (12.5)$$

Notes:

1. **The *restart* command** clears out Maple's memory. This command would be useful after you have performed a large number of calculations and would like to reuse variables in subsequent calculations. Maple can also be re-started by pressing the *restart* button in the toolbar, i.e.,

2. The *evalf*[*10*] command produces a floating-point result with 10 digits. In general, *evalf[n]* will produce a floating-point result with *n* digits, where *n* is an integer.

The following examples show derivatives and integrals, including *double integrals*, for bivariate functions:

diff(*h*(*x*, *y*), *x*)

$$\cos(x\,y)\,y \tag{12.6}$$

diff(*h*(*x*, *y*), *x*$2)

$$-\sin(x\,y)\,y^2 \tag{12.7}$$

$$\frac{\partial}{\partial x}\,g(x,y)$$

$$2\,x \tag{12.8}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\,g(x,y)\right)$$

$$2 \tag{12.9}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\,\mathrm{h}(\mathrm{x},y)\right)$$

$$-\sin(x\,y)\,x\,y+\cos(x\,y) \tag{12.10}$$

$$\int h(x,y)\,\mathrm{d}y$$

$$-\frac{\cos(x\,y)}{x} \tag{12.11}$$

$$\int_0^1 g(x,y)\,\mathrm{d}x$$

$$\frac{1}{3}+y^2 \tag{12.12}$$

$$\int_0^1\int_0^2 h(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

$$\gamma+\ln(2)-Ci(2) \tag{12.13}$$

Notes:

1. The examples above show partial derivatives (12.6) through (12.10), single-variable integration, both indefinite (12.11) and definite (12.12), and a double integral (12.13).
2. The result in (12.13) shows a mathematical constant (γ), and funcions *ln* and *Ci*.

3. The constant γ (gamma) is Euler's constant, its value can be found in Maple by using:

evalf[10](γ);

$$0.5772156649 \quad (12.14)$$

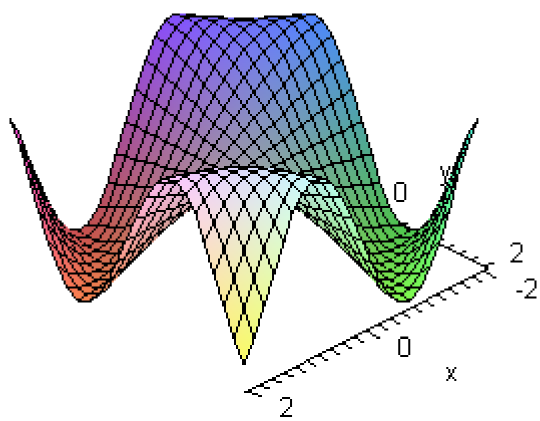
4. The function *ln* is the well-known *natural logarithm* function, and the function *Ci* is referred to as the *cosine integral* function. To find out about this function we can use:

?Ci

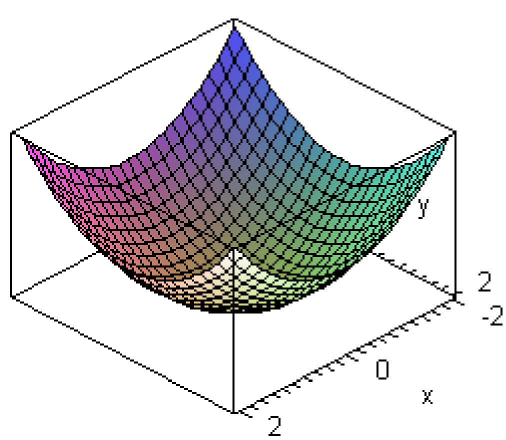
Surface plots for bivariate functions

Function *plot3d* can be used to produce surface plots of functions of the form *f(x,y)*. Some examples are shown below, using the functions *h(x,y)* and *g(x,y)* defined above.

plot3d(*h*(*x*, *y*), *x* = −2 ..2, *y* = −2 ..2, *axes* = *framed*)

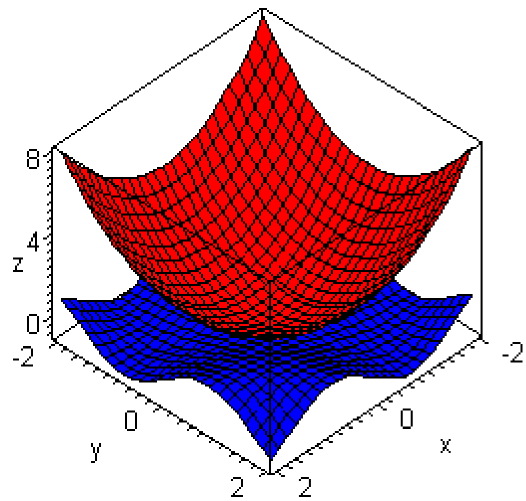


plot3d(*g*(*x*, *y*), *x* = −2 ..2, *y* = −2 ..2, *axes* = *boxed*)



In the following exercise, we produce two plots separately and then use the *plots[display]* command to show the combined plot:

```
p1 := plot3d(h(x, y), x = -2..2, y = -2..2, color = blue) :
p2 := plot3d(g(x, y), x = -2..2, y = -2..2, color = red):
plots[display]( {p1, p2} , axes = boxed, labels =["x","y", "z"])
```

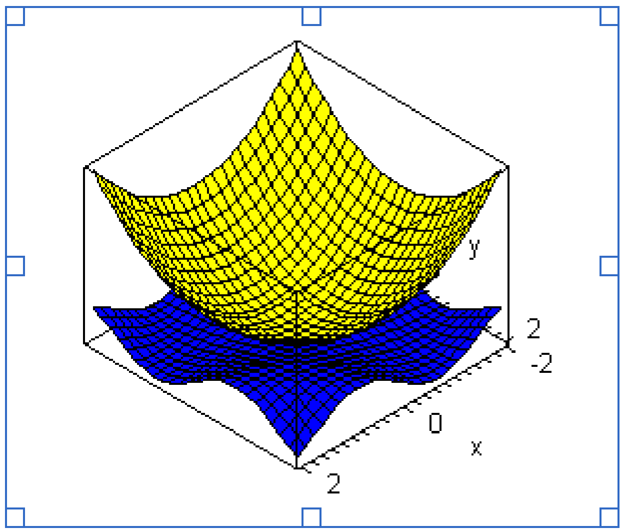


The *plots* package
In some of the plotting examples shown above we have used the command *plots[display]* to display previously-created plots. The command actually combines the name of a package *plots* with the name of the function *display*. An alternative way to activate the function *display* is to load the entire *plots* package with the command *with*.

```
with(plots)
[Interactive, animate, animate3d, animatecurve, arrow, changecoords,            (12.15)
  complexplot, complexplot3d, conformal, conformal3d, contourplot,
  contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display,
  display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, graphplot3d,
  implicitplot, implicitplot3d, inequal, interactive, interactiveparams,
  listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot,
  logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot,
  pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,
  polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d,
  spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d,
  tubeplot]
```

Notice that the output from the command *with(plots)* is a listing of command names, i.e., the names of the commands contained in the *plots* package. [**NOTE**: If you don't want to display the command list, end the *with* command with a colon, i.e., *with*(*plots*):] One of these commands is the *display* command that we used earlier. By using *with(plots)* we load all the commands in that package into Maple's active memory making them available for use. Thus, in the following example, we use the *display* command after loading the *plots* package:

```
with(plots) :
p1 := plot3d(h(x, y), x = -2..2, y = -2..2, color = red) :
p2 := plot3d(g(x, y), x = -2..2, y = -2..2, color = yellow):
display( {p1, p2} , axes = boxed, labels =["x","y", "z"] )
```



Here is another example using the function *animate3d* to see a three-dimensional surface animation after loading *plots* with the command *with*:

```
restart : with(plots) :
h := (x, y, a) → sin(a·x·y)
```

$$(x, y, a) \rightarrow \sin(a\, x\, y) \tag{12.16}$$

```
animate3d(h(x, y, a), x = -2..2, y = -2..2, a = 0..5)
```

Try this exercise in your own Maple worksheet to see the animation used.

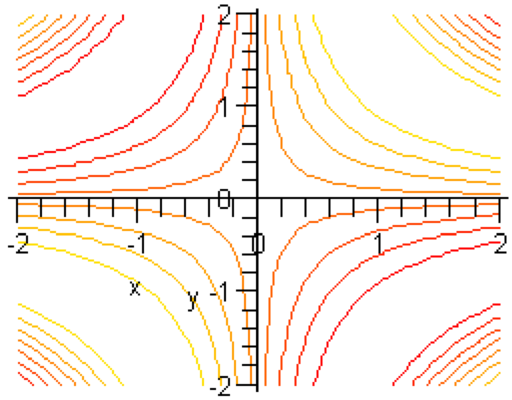
<u>Contour plots and density plot functions</u>

Other plot functions available in the *plots* package for displaying bivariate functions are the functions *contourplot* and *density plot*. Examples of applications of these functions are shown next:

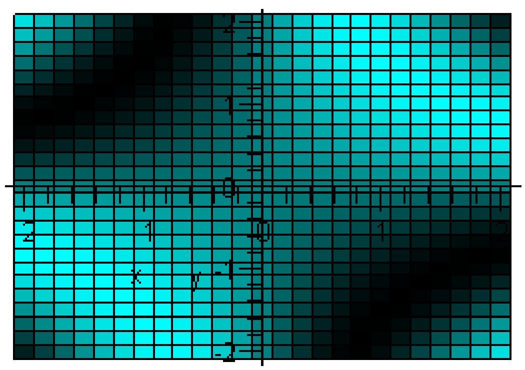
restart : *with*(*plots*) :
$g := (x, y) \rightarrow \sin(x \cdot y)$

$$(x, y) \rightarrow \sin(y\,x) \qquad (12.17)$$

contourplot$(g(x, y), x = -2..2, y = -2..2)$



densityplot$(g(x, y), x = -2..2, y = -2..2, color = cyan)$



Implicit plots

Implicit plots are *x-y* plots out of an equation of the form *f(x,y) = 0*. For example, consider an implicit plot out of a quadratic equation:

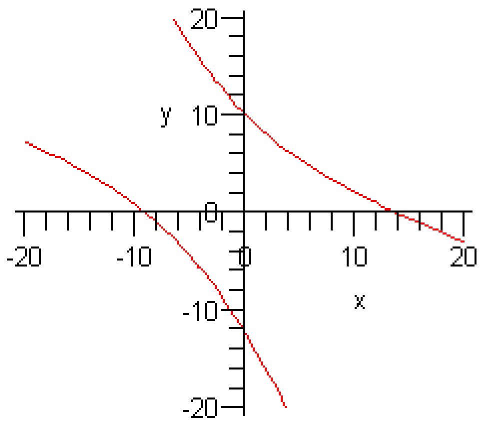
$$x^2 + 3 \cdot x \cdot y + y^2 - 5\,x + 2\,y - 125$$

$restart : with(plots) :$

$eqn1 := x^2 + 3 \cdot x \cdot y + y^2 - 5\,x + 2\,y - 125$

$$x^2 + 3\,x\,y + y^2 - 5\,x + 2\,y - 125 \qquad (12.18)$$

$implicitplot(eqn1, x = -20\,..20, y = -20\,..20, labels = [\text{"x"}, \text{"y"}])$



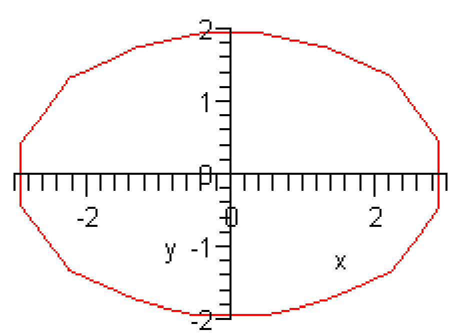
An implicit plot is generated by varying the values of *x* and *y* in a grid on the *xy* plane. You can adjust the *grid* argument to increase the number of points on the plot to produce a smoother graph. The following plots show the effect of changing the grid size:

$restart : with(plots) :$

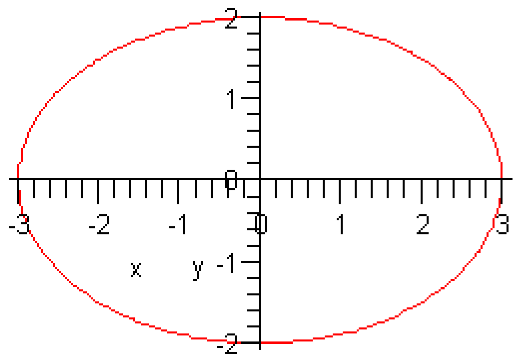
$eqn1 := \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

$$\frac{1}{9}\,x^2 + \frac{1}{4}\,y^2 = 1 \qquad (12.19)$$

$implicitplot(eqn1, x = -4\,..4, y = -4\,..4, grid = [10, 10], scaling = constrained)$



$implicitplot(eqn1, x = -4..4, y = -4..4, grid = [50, 50], scaling = constrained)$



<u>Additional plots from the *plots* package</u>
To find out about additional plots in the *plots* package use *?* followed by the command name, e.g.,

?animate
?animate3d
?implicitplot3d
?logplot
?semilogplot
?loglogplot

As an exercise, try the following plots:

$restart : with(plots) :$
$f := x \to exp(x) + exp(2 \cdot x)$

$$x \to e^x + e^{(2x)} \tag{12.21}$$

$g := x \to ln(x+1)$

$$x \to \ln(x+1) \tag{12.22}$$

$logplot(f(x), x = 0..5, axes = boxed, labels = [\text{"x"},\text{"y"}])$
$semilogplot(g(x), x = 1..1000, axes = boxed, labels = [\text{"x"},\text{"y"}])$
$loglogplot(f(x), x = 1..10, axes = boxed, labels = [\text{"x"}, \text{"y"}])$

(12.23)

Notes:

1. The *logplot* command produces a logarithmic scale in the vertical axis

2. The *semilogplot* command produces a logarithmic scale in the horizontal axis

3. The *loglogplot* command produces logarithmic scales in both axes

Defining multiple-defined functions in Maple
Consider the function defined by

$$f(x) = \begin{cases} x+1 & x < 2 \\ \sqrt{x+1} & x > 2 \end{cases}$$

The *Expression* palette includes the following item for defining a multiple-defined function of one variable with two options, i.e.,

$$\begin{cases} -x & x < 0 \\ x & x > 0 \end{cases}$$

The following example shows the definition of such function and a plot of the same:

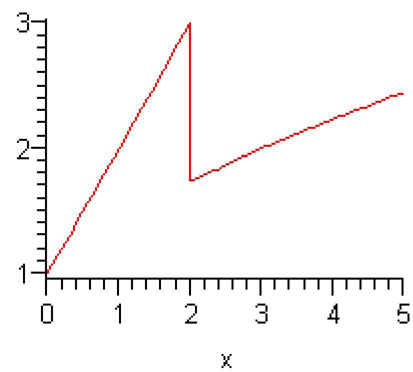
$$f := x \rightarrow \begin{cases} x+1 & x < 2 \\ \sqrt{x+1} & x > 2 \end{cases}$$

$$x \rightarrow \mathit{piecewise}\left(x < 2, x+1, 2 < x, \sqrt{x+1}\right) \qquad (13.1)$$

$f(x)$

$$\begin{cases} x+1 & x < 2 \\ \sqrt{x+1} & 2 < x \end{cases} \qquad (13.2)$$

$\mathit{plot}(f(x), x = 0..5)$

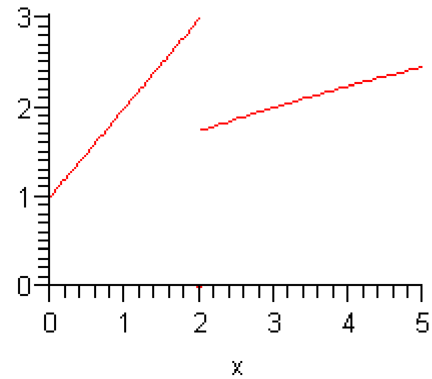


Notes:

1. Use the buttons in the *Expression* palette to define the multiple-defined function
2. Use *f(x)* [Enter] to see the definition of the function
3. The plot shows a discontinuity at *x=2* which is shown as a vertical line.

To eliminate the discontinuity at $x = 2$ you can add the argument *discont = true* to the *plot* command:

$plot(f(x), x = 0..5, discont = true)$



If the function requires more than two definitions simply type the *piecewise* command, as illustrated in the following example:

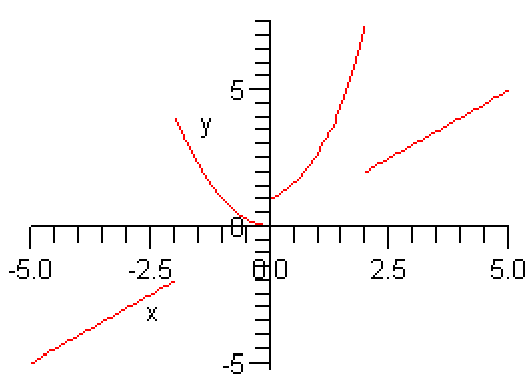
$g := x \rightarrow piecewise(x < -2, x, x < 0, x^2, x < 2, exp(x), x < \infty, x)$

$$x \rightarrow piecewise(x < -2, x, x < 0, x^2, x < 2, e^x, x < \infty, x) \qquad (13.3)$$

$g(x)$

$$\begin{cases} x & x < -2 \\ x^2 & x < 0 \\ e^x & x < 2 \\ x & x < \infty \end{cases} \qquad (13.4)$$

$plot(g(x), x = -5..5, discont = true, labels = ["x", "y"])$



Using units in Maple calculations
In an earlier section we introduced the *Unit Converter* Assistant that allows the user to convert units and place the conversion in a worksheet. In this section we'll explore the use of units within a Maple worksheet.

Maple includes two palettes, the *Units(SI)* and the *Units(FPS)* palettes, that allow the user to attach units to numbers. The contents of the two palettes are shown below:

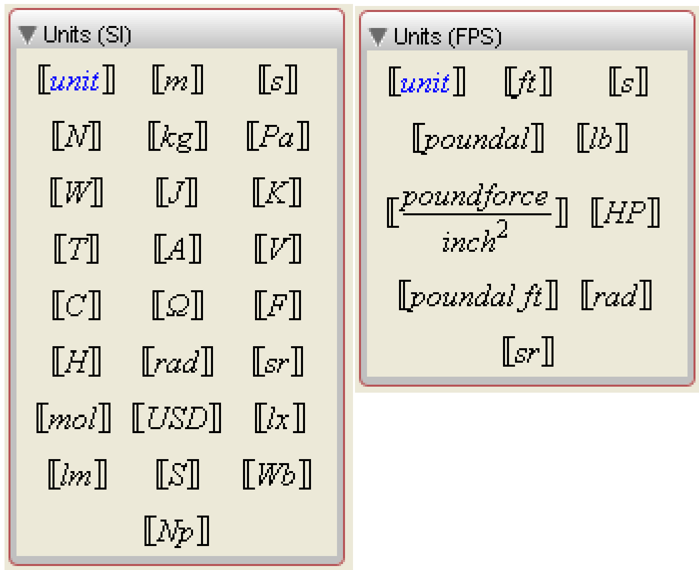


Figure 7 – Unit palettes in Maple interface.

To attach one of the given units, simply type a number in your Maple worksheet and then click on the appropriate unit. The unit references are included in stylized square brackets, as shown in Figure 7. The following is a listing of the units in the *Units(SI)* palette:

- m - meter (length)
- s - second (time)
- N, - newton (force)
- kg - kilogram (mass)
- Pa - pascal (pressure)
- W - watt (power)
- J - joule (work, energy, heat)
- K, - kelvin (absolute temperature)
- T - tesla (magnetic flux density)
- A - ampere (electric current)
- V - volt (potential difference, or voltage)
- C - coulomb (electric charge)
- Ω - ohm (electric resistance)
- F - faraday (electric capacitance)
- H - henry (electric inductance)
- rad - radian (angular measure)
- sr - steradian (solid angle)
- mol - mole (amount of substance)
- USD – U.S. Dollar (money)
- lx - lux (illuminance)
- lm - lumen (luminous flux)
- S - siemens (electric conductance)
- Wb - weber (magnetic flux)
- Np - ??? (???)

The following is a listing of the units available in the *Units (FPS)* palette:

- ft - foot, feet (length)
- s - second (time)
- poundal - poundal (force)
- lb - pound (mass) (*)
- poundforce/inch2 – psi (pressure)
- HP - horsepower (power)
- poundal ft - poundal feet (work, energy,)
- rad - radian (angular measure)
- sr - steradian (solid angle)

Both palettes (*Units(SI)* and *Units(FPS)*) inclued a [[*unit*]] option that allows the user to type any valid combination of units. The following examples show some calculations using units:

First, we do some simple calculation usign units of the SI:

restart :

$$5.5[\![kg]\!]\cdot 12.5\frac{[\![m]\!]}{(8.97[\![s]\!])^2}$$

$$\frac{0.1553549612\,[\![kg]\!]\,[\![m]\!]}{[\![s]\!]^2} \qquad (14.1)$$

$\rightarrow$ 0.1553549612 $[\![N]\!]$

Use context menu (right-click) with option *Units>Simplify* to get the results shown above.

Here is another example:

$$2.5[\![m]\!] + 1.32[\![\frac{m}{s}]\!]\cdot 3.2[\![s]\!] + \frac{1}{2}\cdot 0.75[\![\frac{m}{s^2}]\!]\cdot(3.2[\![s]\!])^2$$

$$2.5\,[\![m]\!] + 4.224\,[\![\frac{m}{s}]\!]\,[\![s]\!] + 3.840000000\,[\![\frac{m}{s^2}]\!]\,[\![s]\!]^2 \qquad (14.2)$$

$\rightarrow$ 10.56400000 $[\![m]\!]$

Once more, use context menu (right-click) with option *Units>Simplify* to get the results shown above.

In most cases, it will be easier to assign values to variables and then operate on a formula, e.g.,*restart*

$$v0 := 2.3[\![\frac{m}{s}]\!] : a := 1.5[\![\frac{m}{s^2}]\!] : x := 12.5[\![m]\!] : x0 := 1.5[\![m]\!] :$$

$v := \sqrt{v0^2 + 2 \cdot a \cdot (x - x0)}$

$$\sqrt{5.29 \, [[\frac{m}{s}]]^2 + 33.00 \, [[\frac{m}{s^2}]] \, [[m]]} \qquad (14.3)$$

$\rightarrow 6.187891402 \, [[\frac{m}{s}]]$

In this case, using the context menu, try *Simplifications>Symbolic* to obtain the result shown above.

NOTE: Make sure to not use unit names, e.g., *m*, *s*, *J*, as variables in your calculations with units.

Here is an example taken from hydraulics:

restart :

$f := 0.012 : L := 100[[ft]] : Diam := 0.5[[ft]] : g := 32.2[[\frac{ft}{s^2}]] : Q := 20[[\frac{ft^3}{s}]] :$

$hf := \frac{8 \cdot f \cdot L \cdot Q^2}{\pi \cdot g \cdot Diam^5}$

$$\frac{10.03922399 \, [[\frac{m^3}{s}]]^2}{[[ft]]^4 \, \pi \, [[\frac{m}{s^2}]]} \qquad (14.4)$$

$\rightarrow 370.2460385 \, [[m]] \rightarrow 1214.717974 \, [[ft]]$

In this case, the option *Simplify>Symbolic* from a context menu in result (14.4) was used to obtain the result 370.2460385 [[*m*]]. Subsequently, the option *Units>Convert >System >FPS* was used to obtain the result in units of the FPS (English or Imperial system).

For additional information on the use of units with Maple, follow this procedure:

- Select *Help* > *Maple Help* (or *Cntl+F1*) to open the Maple help facility
- In the *Search For*: box, click on the *Topic* button, and type *Units* in the text field. Then press [Search]
- There will be a listing of documents on the left-hand side of the Maple interface, click on the document entitled *units(Units).*

This procedure will open a document entitled *Using Units and Dimensions in Maple Documents*. Try the exercises shown in that document to learn more in depth the use of units in Maple 10.

Maple data structures

In this section we present some of the most useful data structures in Maple. These can be used to manipulate data and apply, for example, statistical analysis to data sets. Among the most useful data structures in Maple we find *sequences*, *sets*, *lists*, *vectors*, and *matrices.*

Sequences

A *sequence* in Maple is a collection of objects separated by commas, e.g.:

$seq1 := a, b, c, d$

$$a, b, c, d \tag{15.1}$$

$seq2 := -5, -4, -1, 2$

$$-5, -4, -1, 2 \tag{15.2}$$

Elements of a sequence can be referred to with sub-indices (Shift _), e.g.,

$seq1[1]$

$$a \tag{15.3}$$

$seq2[3]$

$$-1 \tag{15.4}$$

The *seq* (sequence) command can be used to produce sequences if a rule is used to generate the sequence elements, e.g.,

$seq\left(\frac{1}{i^2+1}, i = 0..5\right)$

$$1, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \frac{1}{26} \tag{15.5}$$

$seq3 := seq(sqrt(1+k^2), k = -5..5)$

$$\sqrt{26}, \sqrt{17}, \sqrt{10}, \sqrt{5}, \sqrt{2}, 1, \sqrt{2}, \sqrt{5}, \sqrt{10}, \sqrt{17}, \sqrt{26} \tag{15.6}$$

Sets

When a sequence is enclosed in braces it becomes a *set*. These sets can be operated upon following the rules of set theory. Some examples are shown below:

$A := \{seq(k, k = 0..9)\}$

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \tag{15.4.1}$$

$B := \{seq(2\,k, k = 1..4)\}$

$$\{2, 4, 6, 8\} \tag{15.4.2}$$

$C := \{seq(2\,k+1, k = 1..4)\}$

$$\{3, 5, 7, 9\} \tag{15.4.3}$$

$DD := \{seq(3\,k, k = 1..3)\}$

$$\{3, 6, 9\} \tag{15.4.4}$$

Checking if elements belong to a set, using the function *is* and symbols from the *Common Symbols* palette:

$is(\ 3 \in A)$

$$true \tag{15.4.5}$$

$is(5 \ \in \ B)$

$$false \tag{15.4.6}$$

$is(2 \ \notin C)$

$$true \tag{15.4.7}$$

$is(6 \ \notin \ DD)$

$$false \tag{15.4.8}$$

Checking for subsets using the function *is* and symbols from the *Common Symbols* palette:

$is(B \ \subseteq \ A)$

$$true \tag{15.4.9}$$

$is(DD \ \subseteq B)$

$$false \tag{15.4.10}$$

Unions, interceptions, and Differences using symbols from the *Common Symbols* palette:

$B \cup C$

$$\{2, 3, 4, 5, 6, 7, 8, 9\} \tag{15.4.11}$$

$C \cap DD$

$$\{3, 9\} \tag{15.4.12}$$

$C \setminus DD$

$$\{5, 7\} \tag{15.4.13}$$

Elements of a set, like those of a sequence, can be referred to by using sub-indices (Shift _):

C_3

$$7 \tag{15.4.14}$$

DD_2

$$6 \tag{15.4.15}$$

<u>Lists</u>

Lists are sequences enclosed in square brackets. Many Maple commands operating on a collection of numbers require that those numbers belong to a list. Examples of lists follow.

$$L1 := [seq(s^2 + 1, s = 1..8)]$$

$$[2, 5, 10, 17, 26, 37, 50, 65] \quad (15.5.1)$$

$$L2 := \left[seq\left(\frac{(r-1)}{r+1}, r = 2..5 \right) \right]$$

$$\left[\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3} \right] \quad (15.5.2)$$

Elements of lists can be referenced by their sub-indices, e.g., (brackets can be used to indicate sub-indices, see below)

$$L1[1]$$

$$2 \quad (15.5.3)$$

$$SS := sum(L1[k], k = 1..8)$$

$$212 \quad (15.5.4)$$

$$PP := product(L2[m], m = 1..4)$$

$$\frac{1}{15} \quad (15.5.5)$$

Vectors

Vectors are sequences enclosed between < > brackets. By default, Maple vectors are *column vectors*:

$$U := \langle 5, 3, -2 \rangle$$

$$\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \quad (15.6.1)$$

To enter a row vector, use the command *Vector[row]*([..., ..., ...]), e.g.,

$$V := Vector[row]([6, -1, 2])$$

$$\begin{bmatrix} 6 & -1 & 2 \end{bmatrix} \quad (15.6.2)$$

Elements of a vector can be referred to by sub-indices, e.g.,

$$U[1]$$

$$5 \quad (15.6.3)$$

$$V[3]$$

$$2 \quad (15.6.4)$$

$Z := Vector[row]([8, 2, -1])$

$$\begin{bmatrix} 8 & 2 & -1 \end{bmatrix} \qquad (15.6.6)$$

Some operations that can be performed on vectors correspond to commands in the *LinearAlgebra* package:

$with(LinearAlgebra):$
$Uabs := Norm(U) \quad \# \; The \;\; vector \;\; magnitude$

$$5 \qquad (15.6.7)$$

$DotProduct(U, V)$

$$23 \qquad (15.6.8)$$

$CrossProduct(U, V)$

$$\begin{bmatrix} 4 \\ -22 \\ -23 \end{bmatrix} \qquad (15.6.9)$$

Matrices

Matrices are rectangular array of objects, e.g.,
$M := \langle\langle 3, 1, 5 \rangle | \langle 6, -2, 1 \rangle | \langle 2, -2, 1 \rangle\rangle$

$$\begin{bmatrix} 3 & 6 & 2 \\ 1 & -2 & -2 \\ 5 & 1 & 1 \end{bmatrix} \qquad (15.7.1)$$

$U := \langle 5, -1, 2 \rangle$

$$\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \qquad (15.7.2)$$

$with(LinearAlgebra):$
$Eigenvalues(M)$

$$\begin{bmatrix} 4 \\ -1 + 2\sqrt{3} \\ -1 - 2\sqrt{3} \end{bmatrix} \qquad (15.7.3)$$

$$Eigenvectors(M)$$

$$\left[\begin{bmatrix} 4 \\ -1+2\sqrt{3} \\ -1-2\sqrt{3} \end{bmatrix}, \left[\begin{matrix} [\frac{20}{31}, \frac{4(-5+2\sqrt{3})}{(14+10\sqrt{3})(-4+2\sqrt{3})}, \\ \frac{4(-5-2\sqrt{3})}{(14-10\sqrt{3})(-4-2\sqrt{3})}], [\frac{-7}{31}, -\frac{8+2\sqrt{3}}{14+10\sqrt{3}}, \\ -\frac{8-2\sqrt{3}}{14-10\sqrt{3}}], [1, 1, 1] \end{matrix}\right]\right] \tag{15.7.4}$$

For additional operations with vectors and matrices see the following help entries:

?LinearAlgebra
?VectorCalculus

Data Entry
Date can be entered into Maple by simply typing it into a Maple input, e.g.,

$$restart:$$
$$X := [2.75,\ 1.25,\ 6.23,\ 1.87,\ 4.56,\ 2.31,\ 5.32,\ 2.12, 5.67]$$
$$[2.75, 1.25, 6.23, 1.87, 4.56, 2.31, 5.32, 2.12, 5.67] \tag{16.1}$$

Data can also be entered by reading from a file. Suppose, for example, that the following data is stored in a text file named *Data1.txt*:

```
 1  -2   4  -5
-3   2  -6   3
 8  -1  -7   4
 6  -5   4   2
 1   3  -6  -1
```

We can use the *Import Data* Assistant to read the data as a matrix into Maple. To launch the Assistant use: *Tools* > *Assistants* > *Import Data.*. This will open a dialog box where the user can find the file to open, i.e.,

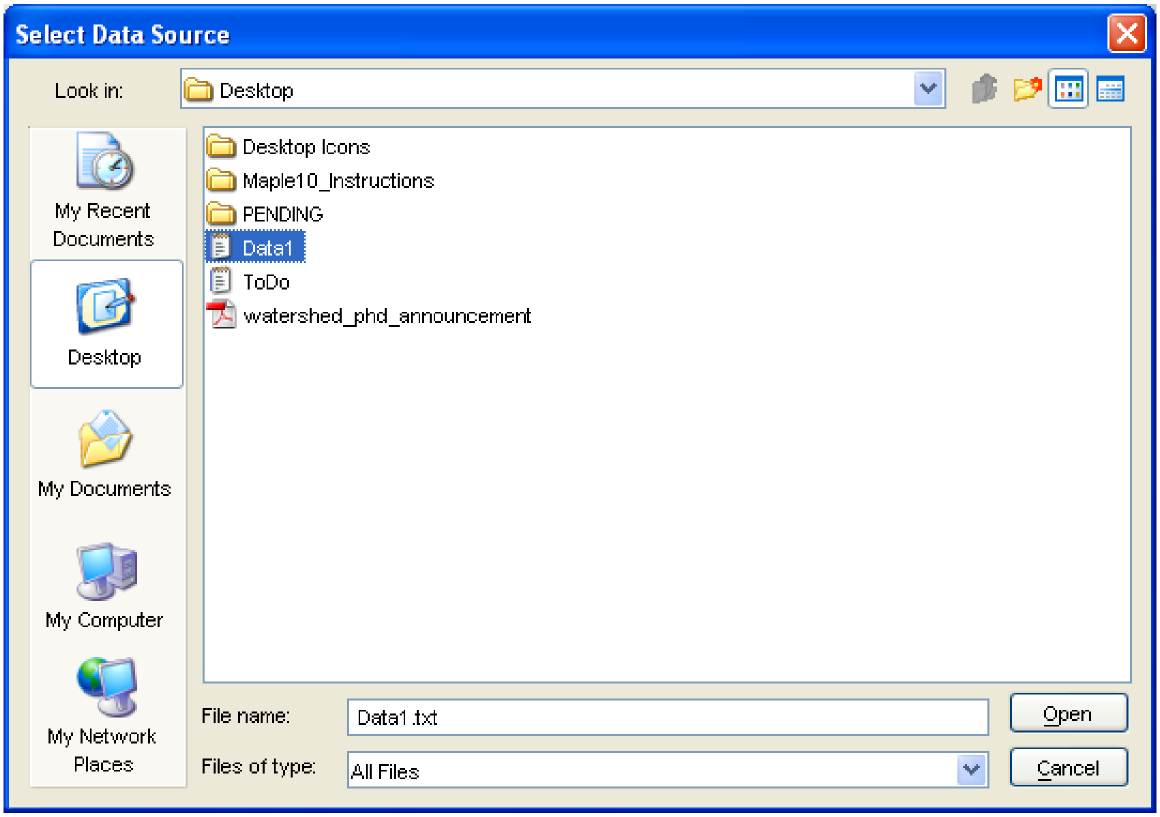


Figure 8 – Dialog box for opening a data file (*Data1.txt*) using the *Import Data...* Assistant.

After pressing [Open], the following dialog box in Figure 9 is shown. This box shows the file in the *View the file* box, indicating that the *Source format* is *Delimited* (i.e., text in columns), and that the data will be assigned to matrix *N* in Maple.

After pressing [OK], Maple will return the following input at the current location of the cursor within the worksheet:

$$N := \begin{bmatrix} 1. & -2. & 4. & -5. \\ -3. & 2. & -6. & 3. \\ 8. & -1. & -7. & 4. \\ 6. & -5. & 4. & 2. \\ 1. & 3. & -6. & -1. \end{bmatrix}$$

After that, you can manipulate the data in the matrix for your own purposes, e.g., spliting the data into columns or rows, calculating statistics, etc.

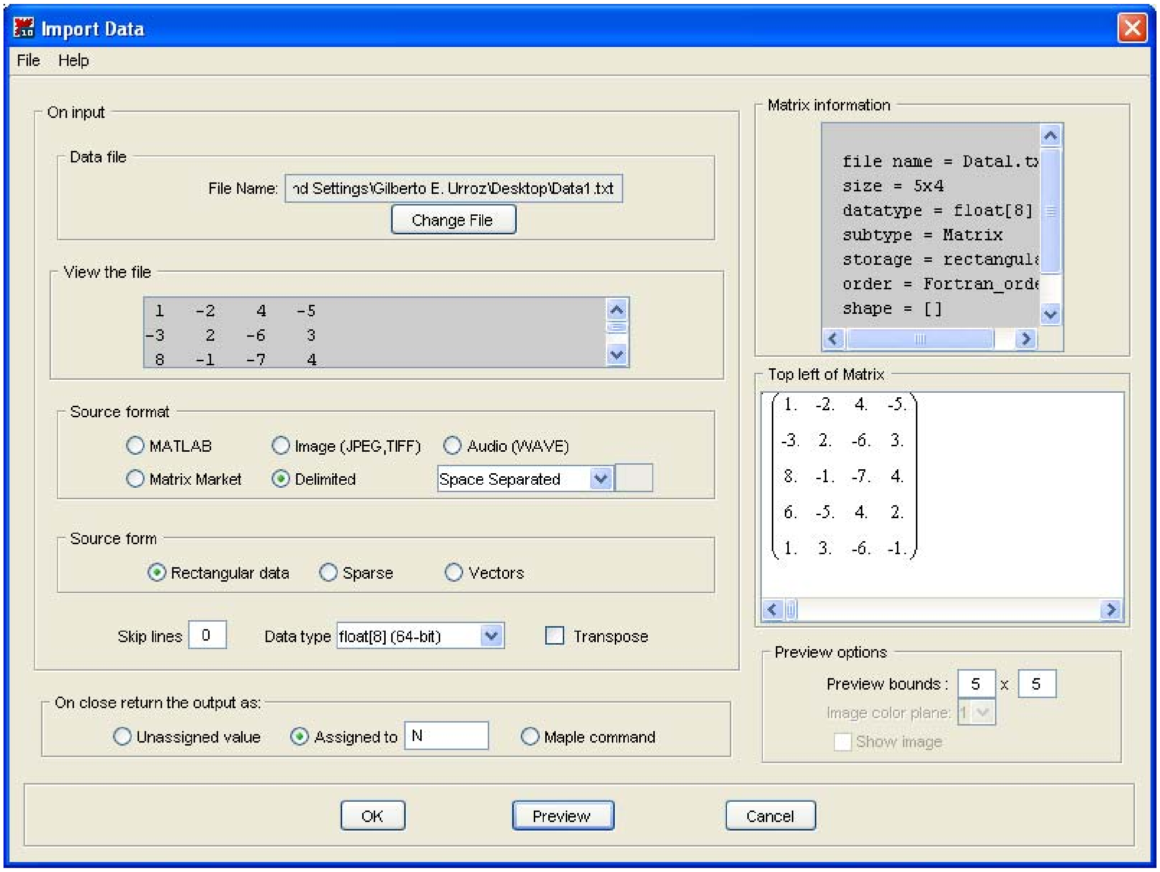


Figure 9 – Import Data dialog box

Solving science and engineering equations

In this section we present some examples of solutions of science and engineering equations, i.e., equations representing physical phenomena.

Example 1 – Single equation

The position of a body in uniformly accelerated motion is given by the equation:

$$x = x_0 + v\,(t\text{-}t_0) + \tfrac{1}{2}\,a(t\text{-}t_o)^2$$

Given the data $x_0 = 2.5\ m$, $v = 1.25\ m/s$, $t_0 = 10.5\ s$, and $a = 6.3\ m/s^2$, determine the time t required to reach a position $x = 25.3\ m$. We can use function *solve* to isolate t from the equation, i.e.,

$restart:$

$eq1 := x = x0 + v\cdot(t - t0) + \frac{1}{2}\cdot a\cdot(t - t0)^2$

$$x = x0 + v\,(t - t0) + \frac{1}{2}\,a\,(t - t0)^2 \qquad (16.1.1)$$

$tsol := solve(eq1, t)$

$$\frac{a\,t0 - v + \sqrt{v^2 + 2\,a\,x - 2\,a\,x0}}{a},\ \frac{a\,t0 - v - \sqrt{v^2 + 2\,a\,x - 2\,a\,x0}}{a} \quad (16.1.2)$$

Next, we define the known variables, including units (use the *Units(SI)* palette):

$x0 := 2.5[[m]]:\ v := 1.25[[\frac{m}{s}]]:\ t0 := 10.5[[s]]:\ a := 6.3[[\frac{m}{s^2}]] : x := 25.3[[m]]:$

and load them into the two results shown in (16.1.2), i.e.,

$t_1 := tsol[1]$

$$\frac{1}{[[\frac{m}{s^2}]]}\left(0.1587301587\left(66.15\,[[\frac{m}{s^2}]]\,[[s]] - 1.25\,[[\frac{m}{s}]] + \sqrt{1.5625\,[[\frac{m}{s}]]^2 + 287.28\,[[\frac{m}{s^2}]]\,[[m]]}\right)\right) \quad (16.1.3)$$

$\rightarrow 12.99926460\,[[s]]$

In (16.1.3) we used a context menu on the result and selected the option *Simplifications > Symbolic* to obtain the result $t_1 = 12.99926460\,s$.

Checking the second solution found in (16.1.2), we have:

$t_2 := tsol[2]$

$$\frac{1}{[[\frac{m}{s^2}]]}\left(0.1587301587\left(66.15\,[[\frac{m}{s^2}]]\,[[s]] - 1.25\,[[\frac{m}{s}]] - \sqrt{1.5625\,[[\frac{m}{s}]]^2 + 287.28\,[[\frac{m}{s^2}]]\,[[m]]}\right)\right) \quad (16.1.4)$$

$\rightarrow 7.603909996\,[[s]]$

Since the motion started at $t_0 = 10.5\ s$, the only reasonable solution is $t = t_1 = 12.99926460\,s$.

Example 2 – System of equations

The motion of a projectile subject to gravity is described the equations

$$x = x_0 + v_0 \cdot cos(\theta_0) \cdot (t - t_0)$$

$$y = y_0 + v_0 \cdot sin(\theta_0) \cdot (t - t_0) - \frac{1}{2} \cdot g \cdot (t - t_0)^2$$

with the y axis in the vertical direction. The projectile is launched from point $P_0(x_0, y_0)$ with a velocity v_0 at an angle θ_0 at time t_0 and is subject to the acceleration of gravity g. The projectile occupies point $P(x, y)$ at time t. In this example, we want to determine the values of the velocity v_0 and the time t required to reach point P if the launch angle θ_0 and the launch location P_0 are known. For this solution we can use function *solve* to solve the two simultaneous equations, i.e.,

restart :

$eqX := x = x0 + v0 \cdot cos(\theta 0) \cdot (t - t0)$

$$x = x0 + v0\cos(\theta 0)\,(t - t0) \tag{16.2.1}$$

$eqY := y = y0 + v0 \cdot sin(\theta 0) \cdot (t - t0) - \frac{1}{2} \cdot g \cdot (t - t0)^2$

$$y = y0 + v0\sin(\theta 0)\,(t - t0) - \frac{1}{2} g\,(t - t0)^2 \tag{16.2.2}$$

$sol := solve(\{eqX, eqY\}, \{v0, t\})$

$$\left\{v0 = -(x - x0) \Big/ \left(\cos(\theta 0)\,\left(-\mathrm{RootOf}(\_Z^2 g\cos(\theta 0) - 2\,\_Z\, g\, t0\cos(\theta 0) + 2y\cos(\theta 0) - 2y0\cos(\theta 0) + g\,t0^2\cos(\theta 0) - 2x\sin(\theta 0) + 2x0\sin(\theta 0)) + t0\right)\right), t = \mathrm{RootOf}(\_Z^2 g\cos(\theta 0) - 2\,\_Z\, g\, t0\cos(\theta 0) + 2y\cos(\theta 0) - 2y0\cos(\theta 0) + g\,t0^2\cos(\theta 0) - 2x\sin(\theta 0) + 2x0\sin(\theta 0))\right\} \tag{16.2.3}$$

The solution *sol* is a set of two elements. This can be figured out by using the function *nops* (*n*umber of *op*erators) on *sol*, i.e.,

$nops(sol)$

$$2 \tag{16.2.4}$$

To separate the solutions we can use:

$sol1 := sol[1]$

$$v0 = -(x - x0) \Big/ \left(\cos(\theta 0)\,\left(-\mathrm{RootOf}(\_Z^2 g\cos(\theta 0) - 2\,\_Z\, g\, t0\cos(\theta 0) + 2y\cos(\theta 0) - 2y0\cos(\theta 0) + g\,t0^2\cos(\theta 0) - 2x\sin(\theta 0) + 2x0\sin(\theta 0)) + t0\right)\right) \tag{16.2.5}$$

$sol2 := sol[2]$

$$t = \mathrm{RootOf}(\_Z^2 g\cos(\theta 0) - 2\,\_Z\, g\, t0\cos(\theta 0) + 2y\cos(\theta 0) - 2y0\cos(\theta 0) + g\,t0^2\cos(\theta 0) - 2x\sin(\theta 0) + 2x0\sin(\theta 0)) \tag{16.2.6}$$

Both *sol1* and *sol2* contain in their expression the function *RootOf*. This function indicates that there is no unique solution on the right-hand side of the equations defining *v0* (in *sol1*) and *t* (in *sol2*). When the function *RootOf* is shown in a result, one can use function *allvalues* to attempt a listing of all possible solutions represented in the *RootOf* rule, e.g.,

$v0sol := allvalues(sol1)$

$$v0 = -(x - x0) \Big/ \left(\cos(\theta 0)\left(-\frac{g\,t0 + \sqrt{-2\,g\,y + 2\,g\,y0 + 2\,g\,x\tan(\theta 0) - 2\,g\,x0\tan(\theta 0)}}{g} + t0\right)\right), v0 = -(x - x0) \Big/ \left(\cos(\theta 0)\left(-\frac{g\,t0 - \sqrt{-2\,g\,y + 2\,g\,y0 + 2\,g\,x\tan(\theta 0) - 2\,g\,x0\tan(\theta 0)}}{g} + t0\right)\right) \quad (16.2.7)$$

$tsol := allvalues(sol2)$

$$t = \frac{g\,t0 + \sqrt{-2\,g\,y + 2\,g\,y0 + 2\,g\,x\tan(\theta 0) - 2\,g\,x0\tan(\theta 0)}}{g}, \quad t = \frac{g\,t0 - \sqrt{-2\,g\,y + 2\,g\,y0 + 2\,g\,x\tan(\theta 0) - 2\,g\,x0\tan(\theta 0)}}{g} \quad (16.2.8)$$

The results shown in (16.2.7) and (16.2.8) indicate that there are two possible values of *v0* and of *t* available, and that, therefore, there are 4 possible solutions to the problem. To list the two values of the variables *v0* and *t* we can use:

$v01 := rhs(v0sol[1])$

$$-(x - x0) \Big/ \left(\cos(\theta 0)\left(-\frac{g\,t0 + \sqrt{-2\,g\,y + 2\,g\,y0 + 2\,g\,x\tan(\theta 0) - 2\,g\,x0\tan(\theta 0)}}{g} + t0\right)\right) \quad (16.2.9)$$

$v02 := rhs(v0sol[2])$

$$-(x - x0) \Big/ \left(\cos(\theta 0)\left(-\frac{g\,t0 - \sqrt{-2\,g\,y + 2\,g\,y0 + 2\,g\,x\tan(\theta 0) - 2\,g\,x0\tan(\theta 0)}}{g} + t0\right)\right) \quad (16.2.10)$$

$t1 := rhs(tsol[1])$

$$\frac{g\,t0 + \sqrt{-2\,g\,y + 2\,g\,y0 + 2\,g\,x\tan(\theta 0) - 2\,g\,x0\tan(\theta 0)}}{g} \qquad (16.2.11)$$

$t2 := rhs(tsol[2])$

$$\frac{g\,t0 - \sqrt{-2\,g\,y + 2\,g\,y0 + 2\,g\,x\tan(\theta 0) - 2\,g\,x0\tan(\theta 0)}}{g} \qquad (16.2.12)$$

If there were numerical values given for the known quantities, at this point we could substitute those values in the expressions for *v01, v02, t1* and *t2*.

Notice that, in the previous solutions, the unknowns (*v0,t*) are relatively easy to isolate because they belong to algebraic terms, i.e., *v0* is first order and *t* is quadratic. A more interesting situation arises if we were to solve for, say, *v0* and θ0, simultaneously, since the terms containing include trigonometric functions (and, therefore, not straightforward to isolate the angle). This is shown next:

$sol := solve(\{eqX, eqY\}, \{v0, \theta 0\})$

$$\Big\{\theta 0 = \arctan\Big((2\,y - 2\,y0 + g\,t^2 - 2\,g\,t\,t0 + g\,t0^2) \Big/ (\mathrm{RootOf}(-g^2\,t0^4 - 8\,y0\,g\,t\,t0 + 4\,y0\,g\,t0^2 + 8\,y\,g\,t\,t0 - 4\,y^2 + 8\,y\,y0 - 4\,y0^2 - g^2\,t^4 - 6\,g^2\,t^2\,t0^2 + 4\,g^2\,t^3\,t0 + 4\,y0\,g\,t^2 - 4\,y\,g\,t0^2 - 4\,y\,g\,t^2 + 4\,g^2\,t\,t0^3 - 4\,x^2 + 8\,x\,x0 - 4\,x0^2 + \_Z^2, label = \_L2)), (2\,x - 2\,x0) \Big/ (\mathrm{RootOf}(-g^2\,t0^4 - 8\,y0\,g\,t\,t0 + 4\,y0\,g\,t0^2 + 8\,y\,g\,t\,t0 - 4\,y^2 + 8\,y\,y0 - 4\,y0^2 - g^2\,t^4 - 6\,g^2\,t^2\,t0^2 + 4\,g^2\,t^3\,t0 + 4\,y0\,g\,t^2 - 4\,y\,g\,t0^2 - 4\,y\,g\,t^2 + 4\,g^2\,t\,t0^3 - 4\,x^2 + 8\,x\,x0 - 4\,x0^2 + \_Z^2, label = \_L2))\Big), v0 = \frac{1}{2\,t - 2\,t0}\,(\mathrm{RootOf}(-g^2\,t0^4 - 8\,y0\,g\,t\,t0 + 4\,y0\,g\,t0^2 + 8\,y\,g\,t\,t0 - 4\,y^2 + 8\,y\,y0 - 4\,y0^2 - g^2\,t^4 - 6\,g^2\,t^2\,t0^2 + 4\,g^2\,t^3\,t0 + 4\,y0\,g\,t^2 - 4\,y\,g\,t0^2 - 4\,y\,g\,t^2 + 4\,g^2\,t\,t0^3 - 4\,x^2 + 8\,x\,x0 - 4\,x0^2 + \_Z^2, label = \_L2))\Big\} \qquad (16.2.13)$$

We suspect there are 2 elements in *sol*, let's check this out:

$nops(sol)$

$$2 \tag{16.2.14}$$

Let's separate the solutions:

$solA := sol[1]$

$$\theta 0 = \arctan\left((2 y - 2 y0 + g t^2 - 2 g t t0 + g t0^2) \big/ (\mathrm{RootOf}(-g^2 t0^4 - 8 y0 g t t0 + 4 y0 g t0^2 + 8 y g t t0 - 4 y^2 + 8 y y0 - 4 y0^2 - g^2 t^4 - 6 g^2 t^2 t0^2 + 4 g^2 t^3 t0 + 4 y0 g t^2 - 4 y g t0^2 - 4 y g t^2 + 4 g^2 t t0^3 - 4 x^2 + 8 x x0 - 4 x0^2 + \_Z^2, label = \_L2)), (2 x - 2 x0) \big/ (\mathrm{RootOf}(-g^2 t0^4 - 8 y0 g t t0 + 4 y0 g t0^2 + 8 y g t t0 - 4 y^2 + 8 y y0 - 4 y0^2 - g^2 t^4 - 6 g^2 t^2 t0^2 + 4 g^2 t^3 t0 + 4 y0 g t^2 - 4 y g t0^2 - 4 y g t^2 + 4 g^2 t t0^3 - 4 x^2 + 8 x x0 - 4 x0^2 + \_Z^2, label = \_L2)) \right) \tag{16.2.15}$$

$solB := sol[2]$

$$v0 = \frac{1}{2 t - 2 t0} (\mathrm{RootOf}(-g^2 t0^4 - 8 y0 g t t0 + 4 y0 g t0^2 + 8 y g t t0 - 4 y^2 + 8 y y0 - 4 y0^2 - g^2 t^4 - 6 g^2 t^2 t0^2 + 4 g^2 t^3 t0 + 4 y0 g t^2 - 4 y g t0^2 - 4 y g t^2 + 4 g^2 t t0^3 - 4 x^2 + 8 x x0 - 4 x0^2 + \_Z^2, label = \_L2)) \tag{16.2.16}$$

$allsolA := allvalues(solA)$

$$\theta 0 = \arctan\left((2 y - 2 y0 + g t^2 - 2 g t t0 + g t0^2) \big/ (g^2 t0^4 + 8 y0 g t t0 - 4 y0 g t0^2 - 8 y g t t0 + 4 y^2 - 8 y y0 + 4 y0^2 + g^2 t^4 + 6 g^2 t^2 t0^2 - 4 g^2 t^3 t0 - 4 y0 g t^2 + 4 y g t0^2 + 4 y g t^2 - 4 g^2 t t0^3 + 4 x^2 - 8 x x0 + 4 x0^2)^{\frac{1}{2}}, (2 x - 2 x0) \big/ (g^2 t0^4 + 8 y0 g t t0 - 4 y0 g t0^2 - 8 y g t t0 + 4 y^2 - 8 y y0 + 4 y0^2 + g^2 t^4 + 6 g^2 t^2 t0^2 - 4 g^2 t^3 t0 - 4 y0 g t^2 + 4 y g t0^2 \right. \tag{16.2.17}$$

$$+ 4\,y\,g\,t^2 - 4\,g^2\,t\,t0^3 + 4\,x^2 - 8\,x\,x0 + 4\,x0^2)^{\frac{1}{2}}\Big), \theta 0 = \arctan\Big($$

$$- (2\,y - 2\,y0 + g\,t^2 - 2\,g\,t\,t0 + g\,t0^2) \Big/ (g^2\,t0^4 + 8\,y0\,g\,t\,t0 - 4\,y0\,g\,t0^2 - 8\,y\,g\,t\,t0 + 4\,y^2 - 8\,y\,y0 + 4\,y0^2 + g^2\,t^4 + 6\,g^2\,t^2\,t0^2 - 4\,g^2\,t^3\,t0 - 4\,y0\,g\,t^2 + 4\,y\,g\,t0^2 + 4\,y\,g\,t^2 - 4\,g^2\,t\,t0^3 + 4\,x^2 - 8\,x\,x0 + 4\,x0^2)^{\frac{1}{2}}, - (2\,x - 2\,x0) \Big/ (g^2\,t0^4 + 8\,y0\,g\,t\,t0 - 4\,y0\,g\,t0^2 - 8\,y\,g\,t\,t0 + 4\,y^2 - 8\,y\,y0 + 4\,y0^2 + g^2\,t^4 + 6\,g^2\,t^2\,t0^2 - 4\,g^2\,t^3\,t0 - 4\,y0\,g\,t^2 + 4\,y\,g\,t0^2 + 4\,y\,g\,t^2 - 4\,g^2\,t\,t0^3 + 4\,x^2 - 8\,x\,x0 + 4\,x0^2)^{\frac{1}{2}}\Big)$$

$allsolB := allvalues(solB)$

$$v0 = \frac{1}{2\,t - 2\,t0}\Big((g^2\,t0^4 + 8\,y0\,g\,t\,t0 - 4\,y0\,g\,t0^2 - 8\,y\,g\,t\,t0 + 4\,y^2 - 8\,y\,y0 + 4\,y0^2 + g^2\,t^4 + 6\,g^2\,t^2\,t0^2 - 4\,g^2\,t^3\,t0 - 4\,y0\,g\,t^2 + 4\,y\,g\,t0^2 + 4\,y\,g\,t^2 - 4\,g^2\,t\,t0^3 + 4\,x^2 - 8\,x\,x0 + 4\,x0^2)^{\frac{1}{2}}\Big), v0 = -\frac{1}{2\,t - 2\,t0}\Big((g^2\,t0^4 + 8\,y0\,g\,t\,t0 - 4\,y0\,g\,t0^2 - 8\,y\,g\,t\,t0 + 4\,y^2 - 8\,y\,y0 + 4\,y0^2 + g^2\,t^4 + 6\,g^2\,t^2\,t0^2 - 4\,g^2\,t^3\,t0 - 4\,y0\,g\,t^2 + 4\,y\,g\,t0^2 + 4\,y\,g\,t^2 - 4\,g^2\,t\,t0^3 + 4\,x^2 - 8\,x\,x0 + 4\,x0^2)^{\frac{1}{2}}\Big) \tag{16.2.18}$$

The results of (16.2.17) and (16.2.18) show closed-form solutions for the values of $\theta 0$ and $v0$. Substituting the values of the known quantities at this point would allow us to find four possible combinations of values of $\theta 0$ and $v0$ for the solution to this problem.

Another problem of interest would be to obtain an equation for the trajectory of the projectile, i.e., an equation of the form $y = f(x)$, if possible. The equations of motion available to us are:

eqX

$$x = x0 + v0\cos(\theta 0)\ (t - t0) \tag{16.2.19}$$

eqY

$$y = y0 + v0\sin(\theta 0)\ (t - t0) - \frac{1}{2} g\ (t - t0)^2 \tag{16.2.20}$$

In order to obtain an expression of the form $y = f(x)$, we need to solve for *(t-t0)* from *eqX* and replace the result in *eqY*, as follows:

$T := solve(eqX, t - t0)$

$$\frac{x - x0}{v0\cos(\theta 0)} \tag{16.2.23}$$

$TrajectoryEqn := subs(t - t0 = T, eqY)$

$$y = y0 + \frac{\sin(\theta 0)\ (x - x0)}{\cos(\theta 0)} - \frac{1}{2}\frac{g\ (x - x0)^2}{v0^2\cos(\theta 0)^2} \tag{16.2.24}$$

Example 3 – Numerical solution to system of equations

The following two equations describe the situation at the entrance from a reservoir into a long open channel with a trapezoidal cross-section. These equations are the energy equation (*eqE*) and Manning's equation (*eqM*):

$$eqE := H = y + \frac{Q^2}{2\cdot g\cdot((b + z\cdot y)\cdot y)^2}$$

$$H = y + \frac{1}{2}\frac{Q^2}{g\ (b + z\,y)^2 y^2} \tag{16.3.1}$$

$$eqM := Q = \frac{Cu}{n}\cdot\frac{((b + z\cdot y)\cdot y)^{\frac{5}{3}}}{\left(b + 2\cdot y\cdot\sqrt{1 + z^2}\right)^{\frac{2}{3}}}\cdot\sqrt{S_0}$$

$$Q = \frac{Cu\ ((b + z\,y)\,y)^{(5/3)}\sqrt{S_0}}{n\left(b + 2\,y\sqrt{1 + z^2}\right)^{(2/3)}} \tag{16.3.2}$$

In these equations, Q is the flow discharge (volume per unit time), Cu is a constant that depends on the system of units used, b is the width of the bottom of the trapezoidal cross-section, y is the depth of flow, z is the side slope of the trapezoidal cross-section's banks ($1H{:}zV$), S_0 is the slope of the channel bed ($1H{:}S_0V$), and n is the Manning's resistance coefficient (which depends on the type of lining of the channel and which can be found in engineering tables).

Manning's equation is dimensionally non-homogeneous, which means that it is not worthy including units in the solution of the equations. However, one must be sure to use a consistent system of units. For this system of equations this means that we should use, for the SI, $Cu = 1.0$, $b(m)$, $y(m)$, $Q(m^3/s)$, and for the English, Imperial, or FPS system, $Cu = 1.486$, $b(ft)$, $y(ft)$, $Q(ft^3/s)$. The quantities n, z, and S_0 are dimensionless (i.e., they have no units attached to them).

In this example we'll use function *fsolve* to obtain a numerical solution to the system on non-linear equations (16.3.1) and (16.3.2) . First, we provide values for the known quantities:

$Cu := 1.486 : \; n := 0.012 : \; b := 3.5 : \; z := 1.5 : \; S_0 := 0.0001 : \; H := 3 : \; g := 32.2 :$

This means $b = 3.5\,ft$, $H = 3.0\,ft$, $g = 32.2\,ft/s^2$. A typical solution requires us to find Q and y. In this case, the units for the solution would be $Q(ft^3/s)$ and $y(ft)$.

After, assigning values to the known quantities, the equations are now:

eqE

$$3 = y + \frac{0.01552795031\, Q^2}{(3.5 + 1.5\, y)^2\, y^2} \tag{16.3.3}$$

eqM

$$Q = \frac{1.238333333\, ((3.5 + 1.5\, y)\, y)^{(5/3)}}{(3.5 + 3.605551276\, y)^{(2/3)}} \tag{16.3.4}$$

A numerical solution is found next:

$sol := fsolve(\{eqE, eqM\}, \{Q, y\})$

$$\{Q = 40.58397325,\ y = 2.953373419\} \tag{16.3.5}$$

Example 4 – Systems of Linear Equations

Systems of linear equations are commonly found in many disciplines. In this example, we consider a system of three linear equations with three unknowns, namely:

$restart :$

$Eq1 := 5 \cdot x + 3 \cdot y + 2 \cdot z = 25$

$$5\,x + 3\,y + 2\,z = 25 \tag{16.4.1}$$

$Eq2 := 2 \cdot x - 6 \cdot y + z = 85$

$$2\,x - 6\,y + z = 85 \tag{16.4.2}$$

$Eq3 := -2 \cdot x + 5 \cdot y + 3 \cdot z = 125$

$$-2\,x + 5\,y + 3\,z = 125 \tag{16.4.3}$$

The simplest way to accomplish a solution would be to use function *solve*:

solve({*Eq1*, *Eq2*, *Eq3*} , {*x*, *y*, *z*})

$$\left\{y=\frac{-1290}{143},\ x=\frac{-1385}{143},\ z=\frac{7185}{143}\right\} \tag{16.4.4}$$

To obtain floating-point values for the solution we can use *evalf*:

evalf((16.4.4))

$$\{y=-9.020979021,\ x=-9.685314685,\ z=50.24475524\} \tag{16.4.5}$$

Alternatively, we could use function *fsolve* on the original system of equations:

fsolve({*Eq1*, *Eq2*, *Eq3*} , {*x*, *y*, *z*})

$$\{y=-9.020979021,\ x=-9.685314685,\ z=50.24475524\} \tag{16.4.6}$$

Systems of linear equations can be solved using matrices. For the system of linear equations under consideration, the matrix equation would be:

$$\begin{bmatrix} 5 & 3 & 2 \\ 2 & -6 & 1 \\ -2 & 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ 85 \\ 125 \end{bmatrix}$$

which is of the form $\mathbf{A} \cdot \xi = \mathbf{b}$, with:

$A := \langle\langle 5, 2, -2\rangle | \langle 3, -6, 5\rangle | \langle 2, 1, 3\rangle\rangle$

$$\begin{bmatrix} 5 & 3 & 2 \\ 2 & -6 & 1 \\ -2 & 5 & 3 \end{bmatrix} \tag{16.4.7}$$

$\xi := \langle x, y, z\rangle$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{16.4.8}$$

$b := \langle 25, 85, 125\rangle$

$$\begin{bmatrix} 25 \\ 85 \\ 125 \end{bmatrix} \tag{16.4.9}$$

To solve for ξ we can use $\xi = \boldsymbol{A}^{-1} \cdot \mathbf{b}$, where $\boldsymbol{A}^{-1}$ is the inverse of matrix $\boldsymbol{A}$. An inverse can be calculated using

A^{-1}

$$\begin{bmatrix} \frac{23}{143} & \frac{-1}{143} & \frac{-15}{143} \\ \frac{8}{143} & \frac{-19}{143} & \frac{1}{143} \\ \frac{2}{143} & \frac{31}{143} & \frac{36}{143} \end{bmatrix} \tag{16.4.10}$$

Thus, the solution to the linear system is:

$\xi := evalm(A^{-1} \&\cdot\ b)$

$$\begin{bmatrix} \frac{-1385}{143} & \frac{-1290}{143} & \frac{7185}{143} \end{bmatrix} \tag{16.4.11}$$

An alternative way to solve this problem is to use function *LinearAlgebra [LinearSolve]*, i.e.,

$\xi := LinearAlgebra[LinearSolve](A, b)$

$$\begin{bmatrix} \frac{-1385}{143} \\ \frac{-1290}{143} \\ \frac{7185}{143} \end{bmatrix} \tag{16.4.12}$$

Some univariate calculus applications

The following are examples of univariate calculus applications. Try the exercises on your own.

<u>Limits</u>

From the *Expression* palette:

restart :

$\lim_{\theta \to 0} \frac{sin(\theta)}{\theta}$

$$1 \tag{17.1.1}$$

Using the full command *limit* for limits from the rightn an from the left:

$$f := x \to \begin{cases} \sqrt{x+1} & x < 2 \\ \dfrac{1}{1+x} & x > 2 \end{cases}$$

$$x \to piecewise\left(x < 2, \sqrt{x+1}, 2 < x, \frac{1}{x+1}\right) \tag{17.1.2}$$

$limit(f(x), x = 2, left)$

$$\sqrt{3} \tag{17.1.3}$$

$limit(f(x), x = 2, right)$

$$\frac{1}{3} \tag{17.1.4}$$

<u>Derivatives as limits</u>

Recall the definition of a derivative:

$$\frac{\mathrm{df}}{\mathrm{dx}} = \lim_{h \to 0} \frac{(f(x+h) - f(x))}{h}$$

Here are a couple of examples of the derivative of functions using this definition as calculated by Maple:

restart

$\lim_{h \to 0} \frac{(f(x+h) - f(x))}{h}$

$$(\mathrm{D}(f))(x) \tag{17.2.1}$$

$\lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h}$

$$\frac{1}{2} \frac{1}{\sqrt{x}} \tag{17.2.1}$$

$\lim_{h \to 0} \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h}$

$$-\frac{1}{x^2} \tag{17.2.2}$$

Note: In (17.2.1), the notation (D(f))(x) is equivalent to df/dx.

Derivative rules

Maple can be used to display derivative rules as indicated in the following examples:

$$\frac{d}{dx} x^n$$

$$\frac{x^n n}{x} \qquad (17.3.1)$$

$$\frac{d}{dx} h(g(x))$$

$$(D(h))(g(x)) \left(\frac{d}{dx} g(x)\right) \qquad (17.3.2)$$

$$\frac{d}{dx} \tan(x)$$

$$1 + \tan(x)^2 \qquad (17.3.3)$$

$$\frac{d}{dx} (h(x) \cdot g(x))$$

$$\left(\frac{d}{dx} h(x)\right) g(x) + h(x) \left(\frac{d}{dx} g(x)\right) \qquad (17.3.4)$$

Notes:

1. Result (17.3.1) is the rule for a power of *x*.
2. Result (17.3.2) is the "chain rule" for composite functions.
3. Result (17.3.3) is the rule for the derivative of the tangent. Typically, it would be given as $sec^2(x)$.
4. Result (17.3.4) is the rule for the derivative of a product.

L'Hopital's rule

L'Hopital's rule is used to evaluate limits of fractions when both numerator and denominator vanish or grow without bound (approach infinity). This exercise illustrates an example of L'Hopital's rule:

$$\lim_{x \to 1} \frac{(x^2 - 1)}{\frac{1}{x} - 1}$$

$$-2 \qquad (17.4.1)$$

$$\lim_{x \to 1} \frac{\left(\frac{d}{dx} (x^2 - 1)\right)}{\frac{d}{dx} \left(\frac{1}{x} - 1\right)}$$

$$-2 \qquad (17.4.2)$$

Determining extrema (maxima and minima) and points of inflection

Given the function

$f := x \to x^3 - 9\,x^2 + 23\,x - 15$

$$x \to x^3 - 9\,x^2 + 23\,x - 15 \tag{17.5.1}$$

To find the extrema we calculate first the first and second derivatives

$fp := x \to \frac{d}{dx} f(x)$

$$x \to \frac{d}{dx} f(x) \tag{17.5.2}$$

$fp(x)$

$$3\,x^2 - 18\,x + 23 \tag{17.5.3}$$

$fpp := x \to \frac{d}{dx} fp(x)$

$$x \to \frac{d}{dx} fp(x) \tag{17.5.4}$$

$fpp(x)$

$$6\,x - 18 \tag{17.5.5}$$

To determine extreme points, we solve $fp(x) = 0$:

$xe := solve(fp(x) = 0, x)$

$$3 + \frac{2}{3}\sqrt{3},\, 3 - \frac{2}{3}\sqrt{3} \tag{17.5.6}$$

Then, we replace the values of the solution xe in the second derivative $fpp(x)$:

$fpp1 := eval(fpp(x), x = xe[1])$

$$4\sqrt{3} \tag{17.5.7}$$

$fpp1 > 0$ means that $xe[1]$ is a local minimum.

$fpp2 := eval(fpp(x), x = xe[2])$

$$-4\sqrt{3} \tag{17.5.8}$$

$fpp1 > 0$ means that $xe[1]$ is a local maximum.

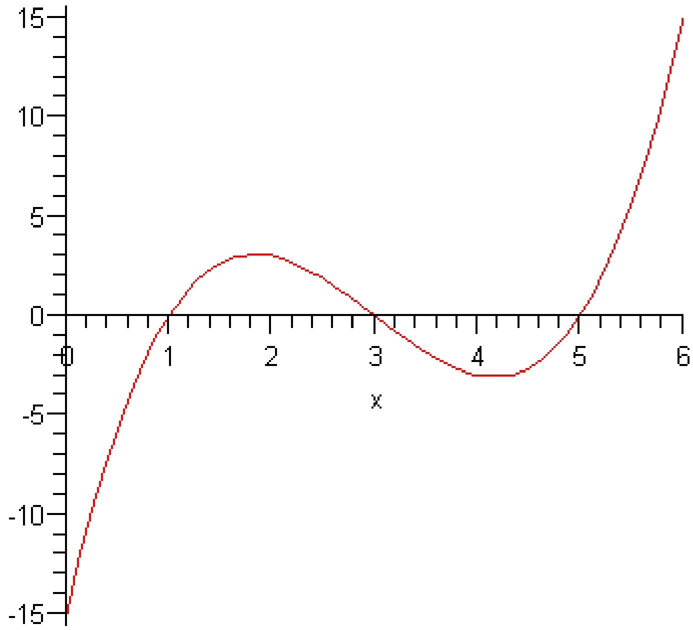
Inflection points are found by solving $fpp(x) = 0$:

$xi := solve(fpp(x) = 0, x)$

$$3 \tag{17.5.9}$$

The plot of the function is shown next:

$plot(f(x), x = 0..6)$



Taylor series expansions

Use function *series* to produce a Taylor series expansion of function *f(x)* about a point $x=a$ including an error term of order n: *series(f(x),x=a,n)*

The definition of Taylor series expansion (error of order 6) is:
restart :

$series(f(x), x = a, 5)$

$$f(a) + (D(f))(a)\,(x-a) + \frac{1}{2}((D^{(2)})(f))(a)\,(x-a)^2 + \frac{1}{6}((D^{(3)})(f))(a)\,(x-a)^3 + \frac{1}{24}((D^{(4)})(f))(a)\,(x-a)^4 + O((x-a)^5) \quad (17.6.1)$$

Other examples of series expansions follow:

$series\left(\sin(x),\ x = \frac{Pi}{2},\ 10\right)$

$$1 - \frac{1}{2}\left(x - \frac{1}{2}\pi\right)^2 + \frac{1}{24}\left(x - \frac{1}{2}\pi\right)^4 - \frac{1}{720}\left(x - \frac{1}{2}\pi\right)^6 + \frac{1}{40320}\left(x - \frac{1}{2}\pi\right)^8 + O\left(\left(x - \frac{1}{2}\pi\right)^{10}\right) \quad (17.6.1)$$

$series(\ln(x),\ x = 1,\ 10)$

$$x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + \frac{1}{7}(x-1)^7 - \frac{1}{8}(x-1)^8 + \frac{1}{9}(x-1)^9 + O\left((x-1)^{10}\right) \quad (17.6.3)$$

Simple solutions to ordinary differential equations

Function *dsolve* can be used to solve differential equations. If a symbolic (closed-form) solution is available, *dsolve* will provide it as the first option. If no symbolic solution is returned, the option *numeric* will allow for a numerical solution. The function *odeadvisor* is useful in classifying equations.

Example 1-First-order, homogeneous

$ode1 := diff(y(x), x) + \sin(x) = 0$

$$\frac{d}{dx}y(x) + \sin(x) = 0 \quad (18.1.1)$$

$ODETools[odeadvisor](ode1)$

$$[\_quadrature] \quad (18.1.2)$$

$dsolve(ode1, y(x))$

$$y(x) = \cos(x) + \_C1 \quad (18.1.3)$$

Example 2-First-order, non-homogeneous

$ode2 := diff(x(t), t) - 2 \cdot x(t) = \sin(t)$

$$\frac{d}{dt}x(t) - 2\,x(t) = \sin(t) \quad (18.2.1)$$

$ODETools[odeadvisor](ode2);$

$$[[\_linear,\ class\ A]] \quad (18.2.2)$$

$dsolve(ode2, x(t))$

$$x(t) = -\frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) + e^{(2t)}\,\_C1 \quad (18.2.3)$$

Example 3 - Second-order, homogeneous

ode3 := diff(y(t), t $ 2) + 2· diff(y(t), t) + y(t) = 0

$$\frac{d^2}{dt^2} y(t) + 2 \left(\frac{d}{dt} y(t) \right) + y(t) = 0 \tag{18.3.1}$$

ODETools[odeadvisor](ode3)

$$[[\_2nd\_order, \_missing\_x]] \tag{18.3.2}$$

dsolve(ode3, y(t))

$$y(t) = \_C1\, e^{(-t)} + \_C2\, e^{(-t)}\, t \tag{18.3.3}$$

Example 4 - Second-order, non-homogeneous

ode4 := diff(u(x), x $ 2) + 2· diff(u(x), x) + u(x) = sin(x) + cos(x)

$$\frac{d^2}{dx^2} u(x) + 2 \left(\frac{d}{dx} u(x) \right) + u(x) = \sin(x) + \cos(x) \tag{18.4.1}$$

ODETools[odeadvisor](ode4)

$$[[\_2nd\_order, \_linear, \_nonhomogeneous]] \tag{18.4.2}$$

dsolve(ode4, u(x))

$$u(x) = e^{(-x)}\, \_C2 + e^{(-x)}\, x\, \_C1 - \frac{1}{2} \cos(x) + \frac{1}{2} \sin(x) \tag{18.4.3}$$

Example 5 - First-order, homogeneous system

odes1 := diff(y(x), x) − y(x) + u(x) = 0

$$\frac{d}{dx} y(x) - y(x) + u(x) = 0 \tag{18.5.1}$$

odes2 := diff(u(x), x) − 3 · y(x) − $\frac{u(x)}{2}$ = 0

$$\frac{d}{dx} u(x) - 3\, y(x) - \frac{1}{2} u(x) = 0 \tag{18.5.2}$$

sols := dsolve({odes1, odes2}, {y(x), u(x)})

sols[1]

$$u(x) = e^{\left(\frac{3}{4} x\right)} \left(\_C1 \sin\left(\frac{1}{4} \sqrt{47}\, x \right) + \_C2 \cos\left(\frac{1}{4} \sqrt{47}\, x \right) \right) \tag{18.5.4}$$

sols[2]

$$y(x) = -\frac{1}{12} e^{\left(\frac{3}{4} x\right)} \Big(-\_C1 \sin\left(\frac{1}{4}\sqrt{47}\, x\right) - \_C1 \cos\left(\frac{1}{4}\sqrt{47}\, x\right)\sqrt{47} - \_C2 \cos\left(\frac{1}{4}\sqrt{47}\, x\right) + \_C2 \sin\left(\frac{1}{4}\sqrt{47}\, x\right)\sqrt{47}\Big) \tag{18.5.5}$$

Example 6 - Second-order, with initial conditions

This example includes initial conditions for the solution

$$ode6 := diff(u(t), t\,\$\,2) + \frac{1}{2} \cdot diff(u(t), t) + \frac{3}{2} \cdot u(t) = sin(t) + cos(t)$$

(18.6.1)

$$ics := u(0) = 1, D(u)(0) = -1$$

$$u(0) = 1,\ (\mathrm{D}(u))(0) = -1 \tag{18.6.2}$$

$$dsolve(\{ode6, ics\}, u(t))$$

$$u(t) = -\frac{11}{23} e^{\left(-\frac{1}{4} t\right)} \sin\left(\frac{1}{4}\sqrt{23}\, t\right)\sqrt{23} + e^{\left(-\frac{1}{4} t\right)} \cos\left(\frac{1}{4}\sqrt{23}\, t\right) + 2\sin(t) \tag{18.6.3}$$

Example 7 - Second-order, non-linear, series solution

$$ode7 := diff(x(t), t\,\$\,2) + x(t) \cdot diff(x(t), t) = sin(t)$$

(18.7.1)

$$ics := x(0) = 1, D(x)(0) = -1$$

$$x(0) = 1,\ (\mathrm{D}(x))(0) = -1 \tag{18.7.2}$$

$$dsolve(\{ode7, ics\}, x(t))$$

$$x(t) = \frac{\mathrm{MathieuCPrime}\left(-1, -1, \frac{1}{2} t\right) + \mathrm{MathieuSPrime}\left(-1, -1, \frac{1}{2} t\right)}{\mathrm{MathieuS}\left(-1, -1, \frac{1}{2} t\right) + \mathrm{MathieuC}\left(-1, -1, \frac{1}{2} t\right)} \tag{18.7.3}$$

$$sol := dsolve(\{ode7, ics\}, x(t), series, order = 10)$$

$$x(t) = \left(1 - t + \frac{1}{2} t^2 - \frac{1}{6} t^3 + \frac{1}{6} t^4 - \frac{1}{10} t^5 + \frac{7}{120} t^6 - \frac{61}{1680} t^7 + \frac{869}{40320} t^8 - \frac{2371}{181440} t^9 + \mathrm{O}(t^{10})\right) \tag{18.7.4}$$

If we wanted to plot this solution we need to convert the series into a polynomial, i.e.,

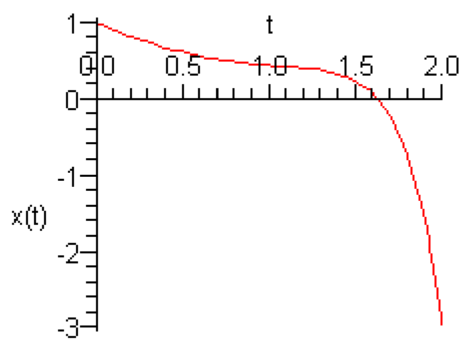
$xp := t \rightarrow convert(rhs(sol), polynom)$

$$t \rightarrow convert(rhs(sol), polynom) \tag{18.7.5}$$

$xp(t)$

$$1 - t + \frac{1}{2} t^2 - \frac{1}{6} t^3 + \frac{1}{6} t^4 - \frac{1}{10} t^5 + \frac{7}{120} t^6 - \frac{61}{1680} t^7 + \frac{869}{40320} t^8 - \frac{2371}{181440} t^9 \tag{18.7.6}$$

$plot(xp(t), t = 0 ..2, labels = ["t","x(t)"])$



Example 8 - Second-order, non-linear, numerical solutions

$restart : ode8 := diff(u(t), t \$ 2) + u(t) \cdot diff(u(t), t) + u(t)^2 = sin(t)$

$$\frac{d^2}{dt^2} u(t) + u(t) \left(\frac{d}{dt} u(t) \right) + u(t)^2 = \sin(t) \tag{18.8.1}$$

$ics := u(0) = 1, D(u)(0) = -2$

$$u(0) = 1, (D(u))(0) = -2 \tag{18.8.2}$$

$sol := dsolve(\{ode8, ics\}, u(t), numeric)$

$$\mathbf{proc}(x\_rkf45) \; ...\mathbf{end\ proc} \tag{18.8.3}$$

The numerical solution, contained in variable *sol*, is actually a Maple procedure (i.e., a program) as indicated in (18.8.2) . To get results out of it we need to invoke the name of the procedure, i.e., *sol*, with an argument representing a value of the independent variable *t*, e.g.,

$sol(2.5)$

```
Error, (in sol) cannot evaluate the solution further
right of 2.3543778, probably a singularity
```

(18.8.4)

The message above indicates that our solution is limited to the range $0<t<2.35$. Let's check the solution for $t = 2.0$:

$sol(2.0)$

$$\left[t=2.0,\ u(t)=-4.53439710339729896,\ \frac{d}{dt}u(t)= -14.8540192107976950\right] \tag{18.8.5}$$

This result indicates that for any value of t, less than $2.35...$, $sol(t)$ returns a list with three equations, the first equation simply repeats the value of t, the second equation shows the value of the solution $u(t)$, and the third value shows the first derivative $u'(t)$.

We can generate lists of values of t and of the second and third components of $sol(t)$,, i. e., $u(t)$ and $u'(t)$, as follows:

$tlist := [seq(0.1 \cdot j, j = 1..23)]$

$$[0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.2, 2.3] \tag{18.8.6}$$

$ulist := [seq(rhs(sol(tlist[j])[2]), j = 1..nops(tlist))]$

$$[0.804983860292817100, 0.619760778182599780, 0.443847825511926086, 0.276513207234011848, 0.116807113286539490, -0.0364353530634958650, -.184616409543046118, -.329425418173025508, -.472923587387860445, -.617676137456614982, -.766954910028067816, -.925050804192115606, -1.09776762559019202, -1.29322945389000710, -1.52325897029131818, -1.80585909740282280, -2.16998728160554100, -2.66554036844046172, -3.38661094511201898, -4.53439710339729896, -6.62932361401094017, -11.5514453138525078, -35.1033812030385732] \tag{18.8.7}$$

$uprimelist := [seq(rhs(sol(tlist[j]))[3]), j = 1..nops(tlist))]$

$$[-1.90062232160106803, -1.80465416346927032, -1.71483319955062963, -1.63344419252151174, -1.56261473532512096, -1.50456306319283817, -1.46185974806718644, -1.43773415151510786, -1.43647167540257481, -1.46398236659529202, -1.52866560500355053, -1.64279678528390560, -1.82486255168524214, -2.10369081833306382, -2.52616571105355758, -3.17259084384706780, -4.18974079796483066, -5.86940494050567806, -8.86045423463932202, -14.8540192107976950, -29.4790337487160912, -81.8603469993630598, -672.602128373557548] \tag{18.8.8}$$

To produce a plot of the solution we can use function *odeplot* in package *plots*, e.g.,

$plots[odeplot](sol, [t, u(t)], t = 0..2, labels = ["t","u(t)"], color = blue)$