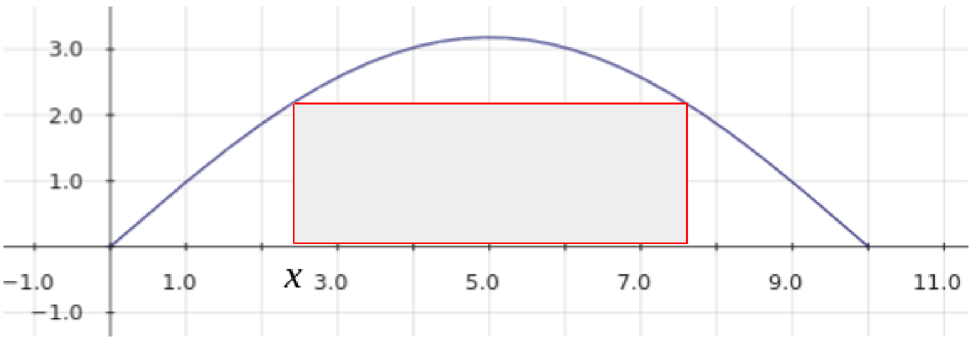
Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

Solution to Assignment #4
Arch and Door
Due on Friday, 12 March.

Architect Rhom builds an arched passage with a flat floor between two buildings, with a cross-section of the arch looking like the graph of $y = \frac{10}{\pi} \sin\left(\frac{\pi x}{10}\right)$ for $0 \leq x \leq 10$.



At the end of the passage Rhom puts a rectangular door whose bottom edge brushes the floor of the passage when it is opened.

1. If Rhom makes the area of the door as large as possible while still fitting inside the arch, what are the height and width of the door? *[10]*

SOLUTION. Suppose the left edge of the door is at x. Then the top edge of the door is part of $y = \frac{10}{\pi} \sin\left(\frac{\pi x}{10}\right)$ and the right edge of the door is at $10 - x$, since the arch is symmetric about the line $x = 5$. This makes the width of the door $(10 - x) - x = 10 - 2x$ and its height x. It follows that the area of the door is given by:

$$A(x) = \frac{10(10 - 2x)}{\pi} \sin\left(\frac{\pi x}{10}\right) = \frac{100 - 20x}{\pi} \sin\left(\frac{\pi x}{10}\right)$$

Note that for practical purposes, the domain of relevant xs is $[0, 5]$. $x \geq 0$ is obvious; $x \leq 5$ follows from the facts that at $x = 5$ we have a door of width 0, and hence area 0, and that for $5 \leq x \leq 10$ we get the same rectangles we got for $x \leq 5$ in reverse order.

At the endpoints of the interval $[0, 5]$ we have

$$A(0) = \frac{100 - 20 \cdot 0}{\pi} \sin\left(\frac{\pi \cdot 0}{10}\right) = \frac{100 - 0}{\pi} \sin(0) = \frac{100}{\pi} \cdot 0 = 0$$

$$\text{and } A(5) = \frac{100 - 20 \cdot 5}{\pi} \sin\left(\frac{\pi \cdot 5}{10}\right) = \frac{100 - 100}{\pi} \sin\left(\frac{\pi}{2}\right) = 0 \cdot 1 = 0\,.$$

It remains to find and check any critical points in the interval $[0,5]$.

$$\begin{aligned}
A'(x) &= \frac{d}{dx}\left(\frac{100-20x}{\pi}\sin\left(\frac{\pi x}{10}\right)\right) \\
&= \left[\frac{d}{dx}\left(\frac{100-20x}{\pi}\right)\right]\cdot\sin\left(\frac{\pi x}{10}\right)+\left(\frac{100-20x}{\pi}\right)\cdot\left[\frac{d}{dx}\sin\left(\frac{\pi x}{10}\right)\right] \\
&= -\frac{20}{\pi}\sin\left(\frac{\pi x}{10}\right)+\left(\frac{100-20x}{\pi}\right)\cos\left(\frac{\pi x}{10}\right)\cdot\left[\frac{d}{dx}\left(\frac{\pi x}{10}\right)\right] \\
&= -\frac{20}{\pi}\sin\left(\frac{\pi x}{10}\right)+\left(\frac{100-20x}{\pi}\right)\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi x}{10}\right) \\
&= -\frac{20}{\pi}\sin\left(\frac{\pi x}{10}\right)+(10-2x)\cos\left(\frac{\pi x}{10}\right)
\end{aligned}$$

Solving $A'(x)=0$ by hand to find the critical points is, at best, likely to be a painful and lengthy experience, so we hand off the problem to `Maple`:

> $eqn := -\left(\frac{20}{\mathrm{Pi}}\right)\cdot\sin\left(\frac{\mathrm{Pi}\cdot x}{10}\right)+(10-2\cdot x)\cdot\cos\left(\frac{\mathrm{Pi}\cdot x}{10}\right)=0:$

> $solve(eqn)$

$$\frac{10\ RootOf\left(2\tan(\_Z)-\pi+2\ \_Z\right)}{\pi} \tag{1}$$

Since `solve` didn't give us an answer that was useful, we try using the `fsolve` command, which tries to find a numerical (and usually approximate) solution.

> $fsolve(eqn)$

$$-5.904082219 \tag{2}$$

This gave an x outside the interval $[0,5]$, which is counterintuitive because there ought to be some critical point in the interval, given that the endpoints give area 0 and there must be rectangles with positive area in between. The problem here is that `fsolve` normally returns only one solution, unless it is given a polynomial equation in one variable, which our equation is not. One can work around this by asking `fsolve` to return a solution in a specified interval:

> $fsolve(eqn, x, 0..5)$

$$2.261473132 \tag{3}$$

Finally, we use `Maple` as a calculator to compute the area corresponding to this critical point:

> $x := 2.261473132$

$$x := 2.261473132 \tag{4}$$

> $\frac{(100-20\cdot x)}{\mathrm{Pi}}\cdot\sin\left(\frac{\mathrm{Pi}\cdot x}{10}\right)$

$$11.37018902 \tag{5}$$

As this is larger than the areas for the endpoints, it's very likely that the maximum area of a rectangular door as specified in the problem is ≈ 11.37 square units.

The question asked us to find the actual dimensions of the door of maximum area. The width of the door is (approximately!) $10 - 2 \cdot 2.261473132 = 5.477053736$ and its height is (approximately!) $\dfrac{10}{\pi} \sin\left(\dfrac{\pi \cdot 2.261473132}{10}\right) = 2.07596813408$ (all in whatever the units are). ■