Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

Solutions to Quiz #4
Tuesday, 9 February.

Do *all three* of the following questions.* Show all your work! Simplify where you conveniently can.

1. Find $\frac{dy}{dx}$ if $y = \arctan\left(\frac{1}{x}\right)$. How is $\frac{dy}{dx}$ related to the derivative of $\arctan(x)$? *[2]*

SOLUTION. We will use the fact that $\frac{d}{dt}\arctan(t) = \frac{1}{1+t^2}$, as well as as the Chain and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\arctan\left(\frac{1}{x}\right) = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{d}{dx}\left(\frac{1}{x}\right) && \text{[Chain Rule]}\\ &= \frac{1}{1+\frac{1}{x^2}}\cdot\frac{d}{dx}\left(x^{-1}\right) = \frac{1}{1+\frac{1}{x^2}}\cdot\left(-x^{-2}\right) && \text{[Power Rule]}\\ &= \frac{1}{1+\frac{1}{x^2}}\cdot\frac{-1}{x^2} = \frac{-1}{x^2+1}\end{aligned}$$

This is the negative of the derivative of $\arctan(x)$. □

NOTE. The fact that $\frac{d}{dx}\arctan\left(\frac{1}{x}\right) = -\frac{d}{dx}\arctan(x)$ is a consequence of a fairly obscure (inverse) trigonometric identity: $\arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} - \arctan(x) & \text{if } x > 0 \\ -\frac{\pi}{2} - \arctan(x) & \text{if } x < 0 \end{cases}$

2. Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y if $\cos(y) = x\sin(y)$. *[2]*

SOLUTION. We throw $\frac{d}{dx}$ at both sides of the given equation and then solve for $\frac{dy}{dx}$. Along the way we'll be using both the Chain and Product Rules.

$$\begin{aligned}\cos(y) = x\sin(y) &\implies \frac{d}{dx}\cos(y) = \frac{d}{dx}\left[x\sin(y)\right]\\ \text{[Chain \& Product Rules]} &\implies -\sin(y)\cdot\frac{dy}{dx} = \left[\frac{dx}{dx}\right]\cdot\sin(y) + x\cdot\left[\frac{d}{dx}\sin(y)\right]\\ \text{[Chain Rule]} &\implies -\sin(y)\frac{dy}{dx} = 1\cdot\sin(y) + x\cos(y)\cdot\frac{dy}{dx}\\ \text{[Rearranging]} &\implies -\sin(y) = \left(\sin(y) + x\cos(y)\right)\frac{dy}{dx}\\ \text{[Solving]} &\implies \frac{dy}{dx} = \frac{-\sin(y)}{\sin(y)+x\cos(y)} \qquad \square\end{aligned}$$

* I was just going to have two questions but then I thought of a connected pair . . . :-)

3. Why do the derivatives $\frac{dy}{dx}$ from questions **1** and **2** have to be equal? *[1]*

SOLUTION. Trying to show they are equal directly is pretty hard, but fortunately there is a much easier way if you solve the equation given in **2** for y.

$$\begin{aligned}
\cos(y) = x\sin(y) &\implies 1 = \frac{x\sin(y)}{\cos(y)} \\
&\implies \frac{1}{x} = \frac{\sin(y)}{\cos(y)} = \tan(y) \\
&\implies \arctan\left(\frac{1}{x}\right) = \arctan\left(\tan(y)\right) = y
\end{aligned}$$

Thus y is the same in **1** and **2** and so $\frac{dy}{dx}$ must be the same, too. ■

[Total = 5]