Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

Solutions to Quiz #3
Tuesday, 2 February.

Do *all three* of the following questions. Show all your work! Simplify where you conveniently can.

1. Compute $\displaystyle\lim_{x\to 0} \sin(x)\sin\left(\frac{1}{x}\right)$. *[2]*

SOLUTION. We will exploit the fact that $0 \le \sin\left(\frac{1}{x}\right) \le 1$ for all $x \ne 0$ and apply the Squeeze Theorem:

$$\begin{array}{rccccc} & 0 & \le & \left|\sin(x)\sin\left(\frac{1}{x}\right)\right| & \le & |\sin(x)| \\ & \downarrow & & \downarrow & & \downarrow \\ \text{As } x \to 0\text{:} & 0 & & ? & & |\sin(0)| = 0 \end{array}$$

Since $0 \le \left|\sin(x)\sin\left(\frac{1}{x}\right)\right| \le |\sin(x)|$ and $\displaystyle\lim_{x\to 0} 0 = 0 = |\sin(0)) = \lim_{x\to 0} |\sin(x)|$, the Squeeze Theorem implies that $\displaystyle\lim_{x\to 0}\left|\sin(x)\sin\left(\frac{1}{x}\right)\right| = 0$, and thus that $\displaystyle\lim_{x\to 0} \sin(x)\sin\left(\frac{1}{x}\right) = 0$. □

2. Use the limit definition of the derivative to compute $\dfrac{d}{dx}\left(\dfrac{x}{x+1}\right)$. *[2]*

SOLUTION. We throw $f(x) = \dfrac{x}{x+1}$ into the limit definition of the derivative and see what happens:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{x+1}\right) &= f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{x+h}{(x+h)+1} - \frac{x}{x+1}}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{x+h}{x+h+1} - \frac{x}{x+1}\right) = \lim_{h\to 0}\frac{1}{h}\cdot\frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)} \\ &= \lim_{h\to 0}\frac{\left(x^2+hx+x+h\right) - \left(x^2+hx+x\right)}{h(x+h+1)(x+1)} = \lim_{h\to 0}\frac{h}{h(x+h+1)(x+1)} \\ &= \lim_{h\to 0}\frac{1}{(x+h+1)(x+1)} = \frac{1}{(x+0+1)(x+1)} = \frac{1}{(x+1)^2} \qquad \square\end{aligned}$$

3. Compute $\dfrac{d}{dx}\left(\dfrac{x}{x+1}\right)$ using the Quotient Rule. *[1]*

SOLUTION. We apply the Quotient Rule and see what happens:

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{\left[\frac{d}{dx}x\right]\cdot(x+1) - x\cdot\left[\frac{d}{dx}(x+1)\right]}{(x+1)^2} = \frac{1\cdot(x+1) - x\cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2} \qquad \blacksquare$$

[Total = 5]