Trent University, Winter 2020

MATH 1120H Test

Thursday, 27 February

Time: 50 minutes

Name: Solutions

Student Number: 0271828

Question	Mark
1	___
2	___
3	___
Total	___ /30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) an aid sheet.

1. Compute any *four* (4) of integrals **a**–**f**. *[12 = 4 × 3 each]*

a. $\displaystyle\int \frac{1}{1-t^2}\,dt$ **b.** $\displaystyle\int_0^1 \arctan(x)\,dx$ **c.** $\displaystyle\int \frac{z^2}{z^3+z}\,dz$

d. $\displaystyle\int_0^\infty ye^{-y^2}\,dy$ **e.** $\displaystyle\int \cos^3(w)\,dw$ **f.** $\displaystyle\int_0^1 \sqrt{1-r^2}\,dr$

SOLUTIONS. **a.** *(Trigonometric substitution)* We will use the substitution $t = \sin(\theta)$, so $dt = \cos(\theta)\,d\theta$. Note that we then have $\cos(\theta) = \sqrt{1-\sin^2(\theta)} = \sqrt{1-t^2}$, $\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{1}{\sqrt{1-t^2}}$, and $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{t}{\sqrt{1-t^2}}$.

$$\begin{aligned}
\int \frac{1}{1-t^2}\,dt &= \int \frac{1}{1-\sin^2(\theta)}\cos(\theta)\,d\theta = \int \frac{\cos(\theta)}{\cos^2(\theta)}\,d\theta = \int \frac{1}{\cos(\theta)}\,d\theta \\
&= \int \sec(\theta)\,d\theta = \ln\left(\sec(\theta)+\tan(\theta)\right) + C \\
&= \ln\left(\frac{1}{\sqrt{1-t^2}} + \frac{t}{\sqrt{1-t^2}}\right) + C = \ln\left(\frac{1+t}{\sqrt{1-t^2}}\right) + C \qquad \blacksquare
\end{aligned}$$

a. *(Partial fractions)* Since $\dfrac{1}{1-t^2}$ has a lower degree in the numerator (0) than the denominator (2), we need not divide the latter into the former. Factoring the denominator, $1-t^2 = (1-t)(1+t)$, so $\dfrac{1}{1-t^2} = \dfrac{A}{1-t} + \dfrac{B}{1+t}$ for some constants A and B. Since

$$\frac{1}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t} = \frac{A(1+t)+B(1-t)}{1-t^2} = \frac{(A-B)t+(A+B)}{1-t^2},$$

we must have $A-B=0$ and $A+B=1$. It follows from the first equation that $A=B$; plugging this into the second equation gives us $A+A=2A=1$, so $A=\dfrac{1}{2}$ and thus $B=\dfrac{1}{2}$, too. This means that

$$\int \frac{1}{1-t^2}\,dt = \int \frac{1/2}{1-t}\,dt + \int \frac{1/2}{1+t}\,dt = \frac{1}{2}\int \frac{1}{1-t}\,dt + \frac{1}{2}\int \frac{1}{1+t}\,dt\,.$$

We will use the respective substitutions $u = 1-t$, so $du = (-1)\,dt$ and $dt = (-1)\,du$, and $w = 1+t$, so $dw = dt$, to handle each of the last two integrals above. Thus

$$\begin{aligned}
\int \frac{1}{1-t^2}\,dt &= \frac{1}{2}\int \frac{1}{1-t}\,dt + \frac{1}{2}\int \frac{1}{1+t}\,dt = \frac{1}{2}\int \frac{1}{u}(-1)\,du + \frac{1}{2}\int \frac{1}{w}\,dw \\
&= -\frac{1}{2}\ln(u) + \frac{1}{2}\ln(w) + C = -\frac{1}{2}\ln(1-t) + \frac{1}{2}\ln(1+t) + C \\
&= \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) + C = \ln\left(\sqrt{\frac{1+t}{1-t}}\right) + C
\end{aligned}$$

Note that this is the same answer, after a little algebra, as was obtained above using a trigonometric substitution. $\blacksquare$

b. *(Integration by parts)* We will use integration by parts, with $u = \arctan(x)$ and $v' = 1$, so $u' = \dfrac{1}{1+x^2}$ and $v = x$. To compute the integral arising from this, we will use the substution $w = 1 + x^2$, so $dw = 2x\,dx$ and $x\,dx = \frac{1}{2}\,dw$.

$$\begin{aligned}
\int_0^1 \arctan(x)\,dx &= x\arctan(x)\Big|_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot x\,dx \\
&= 1\arctan(1) - 0\arctan(0) - \int_{x=0}^{x=1} \frac{1}{w}\cdot\frac{1}{2}\,dw \\
&= \frac{\pi}{4} - 0 - \frac{1}{2}\ln(w)\Big|_{x=0}^{x=1} = \frac{\pi}{4} - \frac{1}{2}\ln\left(1+x^2\right)\Big|_{x=0}^{x=1} \\
&= \frac{\pi}{4} - \left[\frac{1}{2}\ln(2) - \frac{1}{2}\ln(1)\right] = \frac{\pi}{4} - \left[\frac{1}{2}\ln(2) - \frac{1}{2}\cdot 0\right] = \frac{\pi}{4} - \frac{1}{2}\ln(2) \qquad \blacksquare
\end{aligned}$$

c. *(Algebra and substitution)* After simplifying the integrand, we shall use the substitution $w = z^2 + 1$, so $dw = 2z\,dz$ and $z\,dz = \frac{1}{2}\,dw$.

$$\begin{aligned}
\int \frac{z^2}{z^3+z}\,dz &= \int \frac{z^2}{z\left(z^2+1\right)}\,dz = \int \frac{z}{z^2+1}\,dz = \int \frac{1}{w}\cdot\frac{1}{2}\,dw \\
&= \frac{1}{2}\ln(w) + C = \frac{1}{2}\ln\left(z^2+1\right) + C \qquad \blacksquare
\end{aligned}$$

c. *(Partial fractions)* Since $\dfrac{z^2}{z^3+z}$ has a lower degree in the numerator (2) than the denominator (3), we need not divide the latter into the former. Factoring the denominator, $z^3 + z = z\left(z^2+1\right)$, we note that $z^2 + 1 \geq 1 > 0$ for all z, so it has no roots and hence is an irreducible quadratic. It follows that the partial fraction decomposition of the integrand has the form $\dfrac{z^2}{z^3+z} = \dfrac{D}{z} + \dfrac{Ez+F}{z^2+1}$ for some constants D, E, and F. Since

$$\frac{z^2}{z^3+z} = \frac{D}{z} + \frac{Ez+F}{z^2+1} = \frac{D\left(z^2+1\right) + (Ez+F)z}{z\left(z^2+1\right)} = \frac{(D+E)z^2 + Fz + D}{z^3+z},$$

it follows that $D + E = 1$, $F = 0$, and $D = 0$, and hence that $E = 1$. This is fancy way of showing that $\dfrac{z^2}{z^3+z} = \dfrac{z}{z^2+1}$. We may now continue in the same way as in the previous solution after simplifying the integrand. $\blacksquare$

d. Recall that, by definition, $\int_0^\infty ye^{-y^2}\,dy = \lim\limits_{a\to\infty} \int_0^a ye^{-y^2}\,dy$. To evaluate the limit, we first need to compute the definite integral, which we will do with the aid of the substitution $u = -y^2$, so $du = -2y\,dy$ and $y\,dy = -\frac{1}{2}\,du$.

$$\begin{aligned}
\int_0^a ye^{-y^2}\,dy &= \int_{y=0}^{y=a} e^u\left(-\frac{1}{2}\right)\,du = -\frac{1}{2}e^u\Big|_{y=0}^{y=a} = -\frac{1}{2}e^{-y^2}\Big|_{y=0}^{y=a} \\
&= \left(-\frac{1}{2}e^{-a^2}\right) - \left(-\frac{1}{2}e^{-0^2}\right) = \frac{1}{2}\left(1 - e^{-a^2}\right)
\end{aligned}$$

It follows that

$$\int_0^{\infty} ye^{-y^2}\,dy = \lim_{a\to\infty}\int_0^{a} ye^{-y^2}\,dy = \lim_{a\to\infty}\frac{1}{2}\left(1-e^{-a^2}\right) = \frac{1}{2}(1-0) = \frac{1}{2}\,,$$

because as $a \to \infty$, we have $-a^2 \to -\infty$ and hence $e^{-a^2} \to 0$. ■

e. *(Reduction formula)* We will apply the reduction formula for integrating powers of cosine, namely $\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$.

$$\begin{aligned}\int \cos^3(w)\,dw &= \frac{1}{3}\cos^{3-1}(w)\sin(w) + \frac{3-1}{3}\int \cos^{3-2}(w)\,dw\\ &= \frac{1}{3}\cos^2(w)\sin(w) + \frac{2}{3}\int \cos(w)\,dw\\ &= \frac{1}{3}\cos^2(w)\sin(w) + \frac{2}{3}\sin(w) + C \qquad ■\end{aligned}$$

e. *(Trig identity and substitution)* We will use the trigonometric identity $\cos^2(w) + \sin^2(w) = 1$ and the substitution $u = \sin(w)$, so $du = \cos(w)\,dw$.

$$\begin{aligned}\int \cos^3(w)\,dw &= \int \cos^2(w)\cos(w)\,dw = \int \left(1-\sin^2(w)\right)\cos(w)\,dw\\ &= \int \left(1-u^2\right)\,du = u - \frac{u^3}{3} + C = \sin(w) - \frac{1}{3}\sin^3(w) + C\end{aligned}$$

We leave it to the interested reader to check, using suitable trigonometric manipulation, that this answer is actually the same as the one obtained using the reduction formula. ■

f. *(Trig substitution)* We will use the trigonometric substitution $r = \sin(\theta)$, so $dr = \cos(\theta)\,d\theta$ and change the limits as we go along,: $\begin{array}{ccc} r & 0 & 1 \\ \theta & 0 & \pi/2 \end{array}$. We will also use the same cosine reduction formula used in the first solution to **e** above.

$$\begin{aligned}\int_0^1 \sqrt{1-r^2}\,dr &= \int_0^{\pi/2} \sqrt{1-\sin^2(\theta)}\,\cos(\theta)\,d\theta = \int_0^{\pi/2} \sqrt{\cos^2(\theta)}\,\cos(\theta)\,d\theta\\ &= \int_0^{\pi/2} \cos(\theta)\cos(\theta)\,d\theta = \int_0^{\pi/2} \cos^2(\theta)\,d\theta\\ &= \frac{1}{2}\cos^{2-1}(\theta)\sin(\theta)\Big|_0^{\pi/2} + \frac{2-1}{2}\int_0^{\pi/2} \cos^{2-2}(\theta)\,d\theta\\ &= \frac{1}{2}\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\cos(0)\sin(0) + \frac{1}{2}\int_0^{\pi/2} \cos^0(\theta)\,d\theta\\ &= \frac{1}{2}\cdot 0\cdot 1 - \frac{1}{2}\cdot 1\cdot 0 + \int_0^{\pi/2} 1\,d\theta = 0 - 0 - \theta|_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \qquad ■\end{aligned}$$

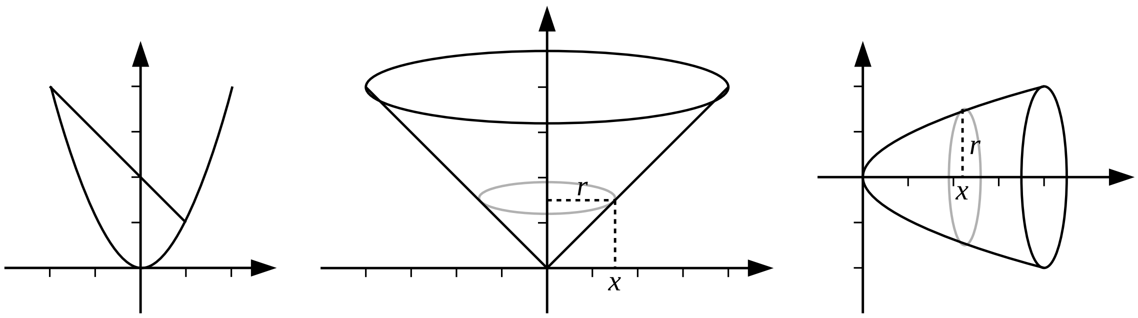
2. Do any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. Find the area of the finite region below $y = 2 - x$ and above $y = x^2$.

b. Find the area of the surface obtained by revolving the curve $y = x$, for $0 \leq x \leq 4$, about the y-axis.

c. Find the volume of the solid obtained by revolving the region between $y = \sqrt{x}$ and $y = 0$, where $0 \leq x \leq 4$, about the x-axis.

SKETCHES. For **a**, **b**, and **c**, respectively:



SOLUTIONS. **a.** We first need to work out where these two curves intersect. They will do so for those values of x where $x^2 = 2-x$, *i.e.* for $x^2+x-2 = 0$. Since $x^2+x-2 = (x+2)(x-1)$, this happens when $x = -2$ or $x = -(-1) = 1$. (If one doesn't spot the factorization, the quadratic equation will give the same answers pretty quickly.) Since the regions between the two curves to the left of $x = -2$ and to the right of $x = 1$ are both infinite in extent, the region in question is the one between $x = -2$ and $x = 1$, for which values of x we have $2 - x \geq x^2$. (For example, $2 - 0 = 2 \geq 0 = 0^2$.) It follows that the area below $y = 2 - x$ and above $y = x^2$ is:

$$\begin{aligned}
\text{Area} &= \int_{-2}^{1} \left(2 - x - x^2\right)\, dx = \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-2}^{1} \\
&= \left(2 \cdot 1 - \frac{1^2}{2} - \frac{1^3}{3}\right) - \left(2 \cdot (-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3}\right) \\
&= \frac{7}{6} - \left(-\frac{10}{3}\right) = \frac{27}{6} = \frac{9}{2} \qquad \blacksquare
\end{aligned}$$

b. First, the increment of arc-length of the curve at x, where $0 \leq x \leq 4$, is given by

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dx}\right)^2}\, dx = \sqrt{1 + 1^2}\, dx = \sqrt{2}\, dx$$

Second, this increment of arc-length is revolved around the y-axis, that is, all the way around a circle of radius $r = x - 0 = x$ and hence circumference $2\pi r = 2\pi x$. We plug these into the integral formula for the area of a surface of revolution:

$$\begin{aligned}
\text{Area} &= \int_0^4 2\pi r \, ds = \int_0^4 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^4 2\pi x \sqrt{2} \, dx = \sqrt{2}\pi \int_0^4 2x \, dx \\
&= \sqrt{2}\pi \cdot x^2 \Big|_0^4 = \sqrt{2}\pi \cdot 16 - \sqrt{2}\pi \cdot 0 = 16\sqrt{2}\pi \qquad \blacksquare
\end{aligned}$$

c. *(Disks/Washers)* Since $y = 0$, better known as the x-axis, is both the lower edge of the region and the axis of rotation, the cross-section of the solid at x, where $0 \le x \le 4$, is a disk of radius $r = y - 0 = \sqrt{x}$, and hence of area $\pi r^2 = \pi \left(\sqrt{x}\right)^2 = \pi x$. These disks are perpendicular to the x-axis, so we should use the variable x in the integral. It follows that the volume of the solid is:

$$\text{Volume} = \int_0^4 \pi r^2 \, dx = \int_0^4 \pi x \, dx = \frac{\pi x^2}{2}\Big|_0^4 = \frac{\pi 16}{2} - \frac{\pi 0}{2} = 8\pi \qquad \blacksquare$$

c. *(Cylindrical Shells)* Since we are revolving the region about the x-axis, cylindrical shell "cross-sections" are parallel to the x-axis and perpendicular to the y-axis, so we need to work in terms of y rather than x. Note that the region between $y = \sqrt{x}$ and $y = 0$ for $0 \le x \le 4$ can also be described as the region between $x = 4$ and $x = y^2$ for $0 \le y \le 2$. The cylindrical shell at y, where $0 \le y \le 2$, has radius $r = y$ and height $h = 4 - y^2$, and hence has area $2\pi rh = 2\pi y \left(4 - y^2\right) = 2\pi \left(4y - y^3\right)$. It follows that the volume of the solid is:

$$\begin{aligned}
\text{Volume} &= \int_0^2 2\pi r h \, dy = \int_0^2 2\pi \left(4y - y^3\right) \, dy = 2\pi \left(2y^2 - \frac{y^4}{4}\right)\Big|_0^2 \\
&= 2\pi \left(2 \cdot 2^2 - \frac{2^4}{4}\right) - 2\pi \left(2 \cdot 0^2 - \frac{0^4}{4}\right) \\
&= 2\pi \cdot (8 - 4) - 2\pi \cdot 0 = 8\pi
\end{aligned}$$

In this particular case, cylindrical shells are a little harder to set up and use than disks. $\blacksquare$

3. Do either *one* (1) of parts **a** or **b**. *[8]*

a. Find the arc-length of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$, where $1 \leq x \leq 2e$.

b. Compute $\displaystyle\int \frac{\cos(x)}{\sin^3(x) + \sin(x)}\, dx$.

SOLUTIONS. **a.** [I'm not even going to try to sketch this curve; if you want to know what it looks like, have a computer plot it for you.] Note that

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{4}x^2 - \frac{1}{2}\ln(x)\right) = \frac{1}{4}\cdot 2x - \frac{1}{2}\cdot\frac{1}{x} = \frac{1}{2}\left(x - \frac{1}{x}\right) .$$

We will plug this and our given limits for x into the arc-length formula and hope for the best.

$$\begin{aligned}
\text{arc-length} &= \int_1^{2e} ds = \int_1^{2e} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^{2e} \sqrt{1 + \left(\frac{1}{2}\left(x - \frac{1}{x}\right)\right)^2}\, dx \\
&= \int_1^{2e} \sqrt{1 + \frac{1}{4}\left(x^2 - 2 + \frac{1}{x^2}\right)}\, dx = \int_1^{2e} \sqrt{\frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2}}\, dx \\
&= \int_1^{2e} \sqrt{\frac{1}{4}\left(x^2 + 2 + \frac{1}{x^2}\right)}\, dx = \int_1^{2e} \sqrt{\frac{1}{4}\left(x + \frac{1}{x}\right)^2}\, dx \\
&= \int_1^{2e} \frac{1}{2}\left(x + \frac{1}{x}\right)\, dx = \frac{1}{2}\left(\frac{x^2}{2} + \ln(x)\right)\Bigg|_1^{2e} \\
&= \frac{1}{2}\left(\frac{4e^2}{2} + \ln(2e)\right) - \frac{1}{2}\left(\frac{1}{2} + \ln(2)\right) \\
&= \left(e^2 + \frac{1}{2}\ln(2) + \frac{1}{2}\ln(e)\right) - \left(\frac{1}{4} + \frac{1}{2}\ln(2)\right) \\
&= e^2 + \frac{1}{2}\ln(2) + \frac{1}{2}\cdot 1 - \frac{1}{4} - \frac{1}{2}\ln(2) = e^2 + \frac{1}{4}
\end{aligned}$$

Not a pretty answer, but it could have been worse. ■

b. We will use the substitution $z = \sin(x)$, so $dz = \cos(x)\,dx$. Then

$$\int \frac{\cos(x)}{\sin^3(x) + \sin(x)}\,dx = \int \frac{1}{z^3 + z}\,dz\,,$$

which last is the integral of a rational function and will be tackled using partial fraction technology. The sharp-eyed may notice that the denominator is the same as the denominator in question **1c** of this test, so the procedure is pretty much the same as in the partial fraction solution to that problem.

Since $\dfrac{1}{z^3 + z}$ has a lower degree in the numerator (0) than the denominator (3), we need not divide the latter into the former. Factoring the denominator, $z^3 + z = z\left(z^2 + 1\right)$, we note that $z^2 + 1 \geq 1 > 0$ for all z, so it has no roots and hence is an irreducible quadratic. It follows that the partial fraction decomposition of the integrand has the form $\dfrac{1}{z^3 + z} = \dfrac{D}{z} + \dfrac{Ez + F}{z^2 + 1}$ for some constants D, E, and F. Since

$$\frac{1}{z^3 + z} = \frac{D}{z} + \frac{Ez + F}{z^2 + 1} = \frac{D\left(z^2 + 1\right) + (Ez + F)z}{z\left(z^2 + 1\right)} = \frac{(D + E)z^2 + Fz + D}{z^3 + z}\,,$$

it follows that $D + E = 0$, $F = 0$, and $D = 1$, and hence that $E = -1$.

Off we go. In one of the resulting integrals we will use the further substitution $u = z^2 + 1$, so $du = 2z\,dz$ and $z\,dz = \frac{1}{2}\,du$.

$$\begin{aligned}
\int \frac{\cos(x)}{\sin^3(x) + \sin(x)}\,dx &= \int \frac{1}{z^3 + z}\,dz = \int \frac{1}{z}\,dz + \int \frac{-z}{z^2 + 1}\,dz \\
&= \ln(z) - \int \frac{1}{u} \cdot \frac{1}{2}\,du = \ln(z) - \frac{1}{2}\ln(u) + C \\
&= \ln(z) - \frac{1}{2}\ln\left(z^2 + 1\right) + C \\
&= \ln\left(\sin(x)\right) - \frac{1}{2}\ln\left(\sin^2(x) + 1\right) + C
\end{aligned}$$

Just to show off a bit, one may rewrite this answer as $\ln\left(\dfrac{\sin(x)}{\sqrt{\sin^2(x) + 1}}\right) + C$. ■

[Total = 30]