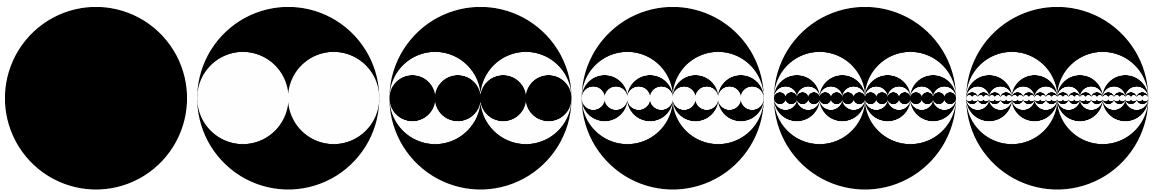
Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solutions to Assignment #4
Alien Eyes?

Consider the following process. At step 0, we start with a solid disk of radius 1. At step 1, we remove two disks of radius $\frac{1}{2}$, which are tangent to each other and to the perimeter of the original disk, from the solid disk. At step 2, we insert two solid disks of radius $\frac{1}{4}$, which are tangent to each other and to the perimeter of the removed disk, into each of the removed disks. At step 3, we remove two disks of radius $\frac{1}{8}$, which are tangent to each other and to the perimeter of the solid disk added at the previous step, from each of the solid disk added at the previous step. In general, at each odd step n we remove two disks of radius $\left(\frac{1}{2}\right)^n$ from each of the solid disks we added at the previous step, and at each even step n we insert two solid disks of radius $\left(\frac{1}{2}\right)^n$ into each of the disks removed at the previous step.

The shapes at steps 0 through 5 are pictured below:



A NOTE ON THE PROCESS. At each step in the process twice as many disks, each of half the radius, are added or subtracted as were subtracted or added at the previous step. Given that we start with a solid disk of radius 1 at step 0, it is not hard to see that it follows that at each step $n > 0$ we add (if n is even) or subtract (if n is odd) 2^n disks, each of radius $\left(\frac{1}{2}\right)^n$, to or from the evolving shape.

1. What is the total length of the perimeter of the (solid) shape at stage $n \geq 0$? *[3]*

SOLUTION. Initially, at step 0, we have a disk of radius 1, and hence perimeter $2\pi r = 2\pi \cdot 1 = 2\pi$. At each step $k > 0$ we either add or subtract 2^k disks of radius $\left(\frac{1}{2}\right)^k$ to or from, respectively, the shape we had a stage $k-1$, thus adding $2^k \cdot 2\pi \left(\frac{1}{2}\right)^k = 2\pi$ to the perimeter of that shape. It follows that after n steps we have a total perimeter of $2\pi + n \cdot 2\pi = 2(n+1)\pi$. □

2. What is the total area of the (solid) shape at stage $n \geq 0$? *[4]*

SOLUTION. Initially, at step 0, we have a disk of radius 1, and hence of area $\pi r^2 = \pi \cdot 1^2 = \pi$. At each step $k > 0$ we either add or subtract 2^k disks of radius $\left(\frac{1}{2}\right)^k$ to or from, respectively, the shape we had a stage $k-1$, thus adding (if k is even) or subtracting (if k is odd) $2^k \cdot \pi \left(\left(\frac{1}{2}\right)^k\right)^2 = \pi \left(\frac{1}{2}\right)^k$ to the area of that shape. It follows that after n steps we have a total area of:

$$\pi - \frac{\pi}{2} + \frac{\pi}{4} - \frac{\pi}{8} + \cdots + \pi\left(-\frac{1}{2}\right)^n = \frac{\pi\left(1-\left(-\frac{1}{2}\right)^{n+1}\right)}{1-\left(-\frac{1}{2}\right)} = \frac{\pi\left(1-\left(-\frac{1}{2}\right)^{n+1}\right)}{\frac{3}{2}}$$
$$= \frac{2}{3}\pi\left(1-\left(-\frac{1}{2}\right)^{n+1}\right)$$

Note that we are using the formula for the sum of a finite geometric series here. This can be found in the text and was mentioned in class, but here is the derivation, just for completeness. Since

$$\begin{aligned}(1-r)\left(a+ar+ar^2+\cdot+ar^{n-1}+ar^n\right) &= a(1-r)\left(1+r+r^2+\cdots+r^{n-1}+r^n\right)\\ &= a\left(1+r+r^2+\cdots+r^{n-1}+r^n\right.\\ &\qquad \left.-r-r^2-\cdots-r^{n-1}-r^n-r^{n+1}\right)\\ &= a\left(1-r^{n+1}\right),\end{aligned}$$

we have that $a+ar+ar^2+\cdot+ar^{n-1}+ar^n = \dfrac{a\left(1-r^{n+1}\right)}{1-r}$, as long as $r \neq 1$. In this particular problem we have $a=\pi$ and $r=-\dfrac{1}{2}$. □

3. What is the total length of the perimeter of the (solid) shape one would have after all infinitely many steps of this process have been done? *[1]*

Solution. Note that the perimeter is only being added to at each stage. The total length of the perimeter of the final shape is the limit of the perimeters of the shapes at the finite stages. Using our formula for the perimeter of the shape at stage n from the solution to **1** and taking the limit, we have $\lim_{n\to\infty} 2(n+1)\pi = \infty$. Thus the final shape, while finite in extent in every direction because it is entirely contained in the original disk, has an infinitely long perimeter. □

4. What is the total area ~~of the perimeter~~ of the (solid) shape one would have after all infinitely many steps of this process have been done? *[2]*

Solution. The total area of the final shape is the limit of the areas of the shapes at the finite stages. Using our formula for the area of the shape at stage n from the solution to **2** and taking the limit, we have $\lim_{n\to\infty} \frac{2}{3}\pi\left(1-\left(-\frac{1}{2}\right)^{n+1}\right) = \frac{2}{3}\pi(1-0) = \frac{2}{3}\pi$ because $\left(-\frac{1}{2}\right)^{n+1} \to 0$ as $n \to \infty$.

The rigorous or paranoid* may object to the last bit in the paragraph. To appease any such, we can appeal to the Squeeze Theorem. Since, as $n \to \infty$, we have

$$\begin{array}{ccccc} -\left(\frac{1}{2}\right)^{n+1} & \leq & \left(-\frac{1}{2}\right)^{n+1} & \leq & \left(\frac{1}{2}\right)^{n+1} \\ \downarrow & & & & \downarrow \\ 0 & & & & 0 \end{array},$$

$\left(-\frac{1}{2}\right)^{n+1}$ has nowhere to go but 0 as $n \to \infty$. ■

* Is there a difference between these in a math class? :-)