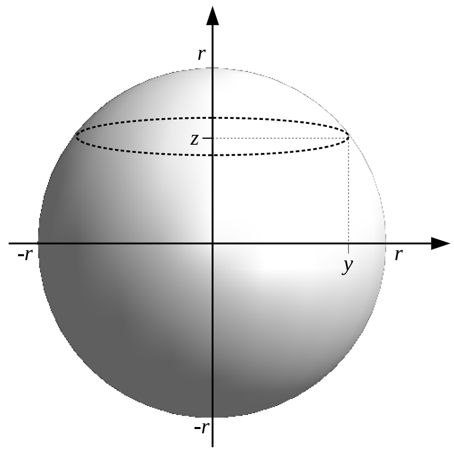
Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2019

Solutions to Assignment #4
Surface ~~Area~~ Error

If you look it up, you will find that the surface area of a sphere of radius r is $SA = 4\pi r^2$. The following calculation comes to a different conclusion.

We will compute the surface area of the sphere by adding up (by integrating) the perimeters of horizontal cross-sections of the sphere, analogously to how we can [correctly!] find the volume of a sphere by adding up (by integrating) the areas of horizontal cross sections of the sphere.

The equation of a sphere of radius r centred at the origin is $x^2 + y^2 + z^2 = r^2$. The cross-section of this sphere for a fixed z with $-r \le z \le r$ is a circle with equation $x^2 + y^2 = r^2 - z^2$ and hence radius $R(z) = \sqrt{r^2 - z^2}$ and perimeter $C(z) = 2\pi R(z) = 2\pi\sqrt{r^2 - z^2}$. Therefore the surface area of the sphere should be $\displaystyle\int_{-r}^{r} C(z)\,dz = \int_{-r}^{r} 2\pi\sqrt{r^2 - z^2}\,dz$. Let's see what happens when we compute this integral. We will substitute $z = r\sin(\theta)$, so $dz = r\cos(\theta)d\theta$, and change the limits as we go along: $\begin{matrix} z & -r & r \\ \theta & -\pi/2 & \pi/2 \end{matrix}$.



$$\begin{aligned}
\int_{-r}^{r} C(z)\,dz &= \int_{-r}^{r} 2\pi\sqrt{r^2 - z^2}\,dz = 2\pi\int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2\sin^2(\theta)}\,r\cos(\theta)\,d\theta \\
&= 2\pi\int_{-\pi/2}^{\pi/2} \sqrt{r^2\cos^2(\theta)}\,r\cos(\theta)\,d\theta = 2\pi\int_{-\pi/2}^{\pi/2} r^2\cos^2(\theta)\,d\theta \\
&= 2\pi r^2\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1 + \cos(2\theta)\right)\,d\theta = \pi r^2\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\pi/2}^{\pi/2} \\
&= \pi r^2\left(\frac{\pi}{2} + \frac{1}{2}\cdot 0\right) - \pi r^2\left(-\frac{\pi}{2} + \frac{1}{2}\cdot 0\right) = \pi^2 r^2
\end{aligned}$$

This is only true in those universes where $\pi = 4$ ...

1. Explain what is wrong in the calculation above. *[4]*

SOLUTION. Read the solution to question **2** below first. The main difference between it and the calculation above is in the setup: the computation above does not take arc-length into account and the solution to **2** below does. A little more generally, the calculation above does not fully account for the slope of the curve being revolved when computing the surface area, while the arc-length based calculation does.

Why does this matter? It matters because the length of a small bit of the curve being revolved matters: the longer it is, that is, the more dy you have *per* dx – *i.e.* the steeper the slope, the more area it contributes when revolved. (See Figure 9.10.4 in the text, and the discussion before it on pages 234-235, for a detailed explanation.) The calculation given above simply doesn't take this fact into account. □

2. Use calculus to compute the surface area of a sphere of radius r correctly. *[6]*

SOLUTION. This is a little easier if we find the surface area of the upper hemisphere of a sphere of radius r first, and then double it to get the surface area of the entire sphere. The upper hemisphere of a sphere of radius r can be obtained by revolving the quarter circle $y = \sqrt{r^2 - x^2}$, for $0 \le x \le r$, about the y-axis.

The formula for the surface area of a surface obtained by revolving a curve $y = f(x)$, $a \le x \le b$, about the y-axis – §9.10 of the textbook, on p. 236 – is:

$$SA = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2}\, dx$$

In our case $f(x) = \sqrt{r^2 - x^2}$, so

$$f'(x) = \frac{d}{dx}\sqrt{r^2 - x^2} = \frac{1}{2\sqrt{r^2 - x^2}} \cdot \frac{d}{dx}\left(r^2 - x^2\right) = \frac{-2x}{2\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}\,,$$

with $a = 0$ and $b = r$. It follows that the surface area of the upper hemispher of radius r is given by:

$$\begin{aligned}
SA &= \int_0^r 2\pi x \sqrt{1 + \left[\frac{-x}{\sqrt{r^2 - x^2}}\right]^2}\, dx = 2\pi \int_0^r x\sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\
&= 2\pi \int_0^r x\sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}}\, dx = 2\pi \int_0^r x\sqrt{\frac{r^2}{r^2 - x^2}}\, dx = 2\pi \int_0^r x\frac{r}{\sqrt{r^2 - x^2}}\, dx \\
&\quad \text{Substitute } u = r^2 - x^2 \text{, so } du = -2x\,dx \text{ and thus } 2x\,dx = (-1)\,du, \\
&\quad \text{and change limits as we go along: } \begin{array}{ccc} x & 0 & r \\ u & r^2 & 0 \end{array} \\
&= \pi r \int_{r^2}^0 \frac{1}{\sqrt{u}}(-1)\,du = \pi r \int_0^{r^2} u^{-1/2}\,du = \pi r \cdot \left.\frac{u^{1/2}}{1/2}\right|_0^{r^2} = 2\pi r \cdot u^{1/2}\Big|_0^{r^2} \\
&= 2\pi r\left(r^2\right)^{1/2} - 2\pi r 0^{1/2} = 2\pi r \cdot r - 0 = 2\pi r^2
\end{aligned}$$

Since the surface area of the upper hemisphere is, by symmetry, half the surface area of the entire sphere, the surface area of the entire sphere is $2 \cdot 2\pi r^2 = 4\pi r^2$. ■