“Completing the Square”

Quadratic expressions like the generic ax^2+bx+c, where $a \neq 0$, are often easier to use or handle if they don't have a linear term bx. The technique of “completing the square” effectively allows us to eliminate the linear term.

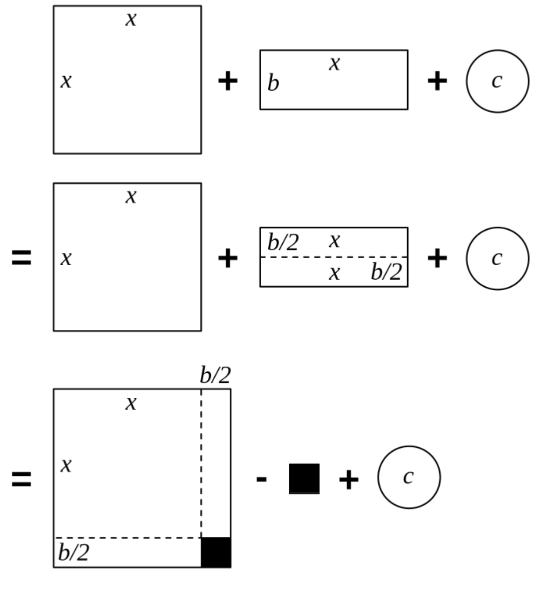
Completing the square in a quadratic expression.

Observe that if we square the expression $x+d$, we get $(x+d)^2 = x^2+2dx+d^2$; in particular, the coefficient of the linear term in the expanded square is twice the constant term in the expression being squared. This is the key to completing the square, which works as follows on the generic ax^2+bx+c:

$$
\begin{aligned}
ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) &= a\left(x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)\\
&= a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right) = a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c\\
&= az^2+\left(c-\frac{b^2}{4a}\right),
\end{aligned}
$$

where the quadratic expression in the new variable $z = x+\dfrac{b}{2a}$ lacks a linear term. The cost to this simplification is having to do the algebra, and then deal with the new variable and the modified constant term.

The process above can be visualized as an area calculation, which is where the expression “completing the square” probably comes from. For example, it can be pictured as follows with $a=1$:



Applications of this trick include finding the locations of the tip of a parabola and its x-intercepts, deriving the quadratic formula, and simplifying quadratic expressions in integrals to set up substitutions.

Finding the tip and intercepts of a parabola.

Consider the generic parabola $y = ax^2 + bx + c$. If $a > 0$, then, after completing the square,

$$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

is as small as possible when $\left(x + \frac{b}{2a}\right)^2 = 0$, *i.e.* when $x = -\frac{b}{2a}$. (Since otherwise $a\left(x + \frac{b}{2a}\right)^2 > 0$.) Similarly, if $a < 0$, then

$$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

is as large as possible when $\left(x + \frac{b}{2a}\right)^2 = 0$, *i.e.* when $x = -\frac{b}{2a}$. (Since otherwise $a\left(x + \frac{b}{2a}\right)^2 < 0$.) Either way, this means that the tip of the parabola occurs when $x = -\frac{b}{2a}$, at which point we have $y = -\frac{b^2}{4a} + c$.

Thus the tip of the parabola is at $\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$.

Finding the y-intercept of the parabola $y = ax^2 + bx + c$ is very easy: simply plug in $x = 0$. Thus the y-intercept of the parabola is $y = a0^2 + b0 + c = c$. No need to complete the square here!

To find the x-intercepts of the parabola, we need to find the values of x such that $y = ax^2 + bx + c = 0$. We usually do this by applying the quadratic formula, which tells us that these values of x, if any, are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the *discriminant*, $b^2 - 4ac$, is negative, there are no such real numbers x, so there are no x-intercepts, *i.e.* the entire parabola is above the x-axis or below the x-axis. Note that if the discriminant is positive, then the x-coordinate of the tip of the parabola is exactly halfway between the x-intercepts:

$$\frac{\frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a}}{2} = \frac{\frac{(-b) + (-b)}{2a}}{2} = \frac{-2b}{4a} = -\frac{b}{2a}$$

What does all this have to do with completing the square? Guess where does the quadratic formula come from ...

Deriving the quadratic formula by completing the square.

THEOREM. If $ax^2 + bx + c = 0$, where $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

PROOF. The general idea, of course, is to isolate x.

$$
\begin{aligned}
ax^2 + bx + c = 0 &\implies x^2 + \frac{b}{a}x + \frac{c}{a} = \frac{0}{a} = 0 && \text{Dividing both sides by } a\text{, which we can do safely because } a \neq 0. \\
&\implies x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 && \text{Setting up and } \ldots \\
&\implies \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 && \ldots \text{ completing the square.} \\
&\implies \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2} \\
&\implies x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
&\implies x = -\frac{b}{2a} + \frac{\pm\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \blacksquare
\end{aligned}
$$

Simplifying quadratic expressions in integrands

REFERENCE

1. *Calculus* (Third Edition), by Michael Spivak, Publish or Perish, 1994. ISBN 0-914098-89-6