Trent University, Summer 2026 (S62)

MATH 1110H Midterm Test

10:50–11:50 a.m. on Tuesday, 7 July.

Name: Hero Zero

Student Number: 0000000

Question	Mark	
1	____	
2	____	
3	____	
Total	____	/30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and all sides of one letter- or A4-size aid sheet.
- If you do more than the minimum number of parts or questions, only the first ones the marker finds will be marked. Cross out anything you do not want marked.

1. Do any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. Use the ε–δ definition of limits to check that $\lim_{x \to -1} (4x+3) = -1$.

b. Compute $\lim_{x \to \infty} \frac{x}{\ln(x)}$. **c.** Compute $\lim_{x \to 4} \frac{x^2 - 16}{x^2 - 2x - 8}$.

SOLUTIONS. **a.** According to the ε–δ definition of limits we need to check that for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x - (-1)| < \delta$, then $|(4x+3) - (-1)| < \varepsilon$.

Supppose we are given an $\varepsilon > 0$. We try to reverse-engineer a corresponding $\delta > 0$:

$$\begin{aligned} |(4x+3) - (-1)| < \varepsilon &\iff |4x+3+1| < \varepsilon \iff |4x+4| < \varepsilon \\ &\iff |4(x+1)| < \varepsilon \iff 4|x+1| < \varepsilon \\ &\iff |x - (-1)| < \frac{\varepsilon}{4} \end{aligned}$$

It follows that setting $\delta = \frac{\varepsilon}{4}$ will do. For if $|x - (-1)| < \delta = \frac{\varepsilon}{4}$, we can run the chain of reasoning above in reverse – every step being fully reversible – to obtain $|(4x+3) - (-1)| < \varepsilon$, as required.

Thus $\lim_{x \to -1} (4x+3) = -1$ by the ε–δ definition of limits. $\square$

b. Here we go:

$$\begin{aligned} \lim_{x \to \infty} \frac{x \quad \to \infty}{\ln(x) \to \infty} &= \lim_{x \to \infty} \frac{\frac{d}{dx} x}{\frac{d}{dx} \ln(x)} \quad \text{by l'Hôpital's Rule} \\ &= \lim_{x \to \infty} \frac{1}{\frac{1}{x}} = \lim_{x \to \infty} x = \infty \quad \square \end{aligned}$$

c. *Algebra.* Note that $x = 4$ makes both the numerator and the denominator 0, so we can factor out $x - 4$ in each one to simplify the expression before trying to evaluate the limit.

$$\lim_{x \to 4} \frac{x^2 - 16}{x^2 - 2x - 8} = \lim_{x \to 4} \frac{(x-4)(x+4)}{(x-4)(x+2)} = \lim_{x \to 4} \frac{x+4}{x+2} = \frac{4+4}{4+2} = \frac{8}{6} = \frac{4}{3} \quad \square$$

c. *l'Hôpital's Rule.* Here we go:

$$\begin{aligned} \lim_{x \to 4} \frac{x^2 - 16 \quad \to 4^2 - 16 \qquad = 0}{x^2 - 2x - 8 \to 4^2 - 2 \cdot 4 - 8 = 0} &= \lim_{x \to 4} \frac{\frac{d}{dx}\left(x^2 - 16\right)}{x \frac{d}{dx}\left(x^2 - 2x - 8\right)} \quad \text{by l'Hôpital's Rule} \\ &= \lim_{x \to 4} \frac{2x}{2x - 2} = \frac{2 \cdot 4}{2 \cdot 4 - 2} = \frac{8}{6} = \frac{4}{3} \quad \blacksquare \end{aligned}$$

2. Find $\frac{dy}{dx}$ as best you can in any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. $y = \arctan\left(1/\sqrt{x}\right)$ **b.** $y = \dfrac{x}{x^2 - 2x + 1}$ **c.** $e^{xy} = 41$

SOLUTIONS. **a.** *Chain Rule, etc.* Here goes:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\arctan\left(1/\sqrt{x}\right) = \frac{d}{dx}\arctan\left(x^{-1/2}\right) = \frac{1}{1+\left(x^{-1/2}\right)^2}\cdot\frac{d}{dx}x^{-1/2} \\ &= \frac{1}{1+x^{-1}}\cdot\left(-\frac{1}{2}\right)x^{-3/2} = \frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{2x^{3/2}} = \frac{-1}{2x^{3/2}+2x^{1/2}} = \frac{-1}{2(x+1)\sqrt{x}} \quad \square\end{aligned}$$

b. *Quotient Rule, etc.* Here goes:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x}{x^2-2x+1}\right) = \frac{\left[\frac{d}{dx}x\right]\left(x^2-2x+1\right) - x\left[\frac{d}{dx}\left(x^2-2x+1\right)\right]}{\left((x-1)^2\right)^2} \\ &= \frac{[1]\left(x^2-2x+1\right) - x\,[2x-2]}{(x-1)^4} = \frac{(x-1)^2 - 2x(x-1)}{(x-1)^4} = \frac{(x-1)-2x}{(x-1)^3} \\ &= \frac{-x-1}{(x-1)^3} = -\frac{x+1}{(x-1)^3} \quad \square\end{aligned}$$

c. *Solve for y first.* $e^{xy} = 41 \implies xy = \ln\left(e^{xy}\right) = \ln(41) \implies y = \dfrac{\ln(41)}{x}$. It follows that $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\ln(41)}{x}\right) = \ln(41)\dfrac{d}{dx}x^{-1} = \ln(41)(-1)x^{-2} = -\dfrac{\ln(41)}{x^2}$. $\square$

c. *Implicit differentiation.* Here we go:

$$\begin{aligned}0 &= \frac{d}{dx}41 = \frac{d}{dx}e^{xy} = e^{xy}\frac{d}{dx}(xy) = e^{xy}\left(\left[\frac{d}{dx}x\right]y + x\left[\frac{d}{dx}y\right]\right) \\ &= e^{xy}\left([1]y + x\left[\frac{dy}{dx}\right]\right) = e^{xy}\left(y + x\frac{dy}{dx}\right)\end{aligned}$$

Since $e^t > 0$ for all t, it follows that $y + x\dfrac{dy}{dx} = 0$. Solving this for $\dfrac{dy}{dx}$ gives us $\dfrac{dy}{dx} = -\dfrac{y}{x}$. To get any further, if we needed to, we'd have to solve for y in the original equation, just as we did in the previous solution ... $\square$

3. Do *one* (1) of parts **a** or **b**. *[10]*

a. Find the domain as well as any and all intercepts, horizontal and vertical asymptotes, intervals of increase and decrease, local maximum and minimum points, intervals of concavity, and inflection points of $f(x) = \dfrac{1}{x^2+1}$, and sketch its graph based on this information.

b. Determine the maximum possible area of a rectangle with two corners on the x-axis and the other two corners on the part of the parabola $y = 4 - x^2$ that is above the x-axis.

SOLUTIONS. **a.** We run through the indicated checklist:

i. Domain. Since $x^2 + 1 \geq 1 > 0$ for all x, $f(x) = \dfrac{1}{x^2+1}$ is defined for all x. Note that it is contnuous and differentiable where it is defined (*i.e.* everywhere) because it is a rational function.

ii. Intercepts. $f(0) = \dfrac{1}{0^2+1} = 1$, so the y-intercept is at $y = 1$.

Since $1 \neq 0$, $f(x) = \dfrac{1}{x^2+1} \neq 0$ for all x; that is, there is no x-intercept.

iii. Vertical asymptotes. Since $f(x)$ is defined and continuous for all x, as previously noted, $y = f(x)$ cannot have any vertical asymptotes.

iv. Horizontal asymptotes. We check the behaviour of the function in both directions. Note that as $x \to \pm\infty$, $x^2 \to +\infty$.

$$\lim_{x\to-\infty} \frac{1}{x^2+1} \begin{matrix} \to 1 \\ \to +\infty \end{matrix} = 0^+ \quad \text{and} \quad \lim_{x\to+\infty} \frac{1}{x^2+1} \begin{matrix} \to 1 \\ \to +\infty \end{matrix} = 0^+.$$

Thus $y = f(x)$ has a horizontal asymptote of $y = 0$ in both directions, which it approaches from above in each case.

v. Increase/decrease/max/min. We first need to compute $f'(x)$. We could use the Quotient Rule, but this time we will go with a combination of the Power and Chain Rules instead.

$$\begin{aligned} f'(x) &= \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^2+1}\right) = \frac{d}{dx}\left(x^2+1\right)^{-1} = (-1)\left(x^2+1\right)^{-2}\frac{d}{dx}\left(x^2+1\right) \\ &= \frac{-1}{\left(x^2+1\right)^2} \cdot 2x = \frac{-2x}{\left(x^2+1\right)^2} \end{aligned}$$

Note that $\left(x^2+1\right)^2 > 0$ for all x. It follows that $f'(x)$ is defined (and continuous and differentiable) everywhere. Also, $f'(x)$ is negative, zero, or positive exactly as the numerator, $-2x$, is negative, zero, or positive, which happens when $x > 0$, $x = 0$, and $x < 0$, respectively. Thus $f(x)$ is decreasing when $x > 0$ and increasing when $x < 0$, which makes $x = 0$ a maximum. We summarize this in the usual table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	$\uparrow$	max	$\downarrow$

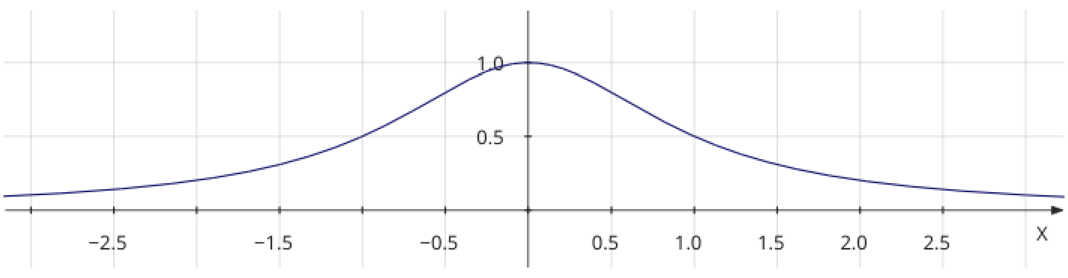
vi. Concavity and inflection. We need to compute $f''(x)$. We could use a combination of the Power, Product, and Chain Rules, but this time we will use the Quotient Rule as our principal tool.

$$\begin{aligned}
f''(x) &= \frac{d^2y}{dx^2} = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{-2x}{\left(x^2+1\right)^2}\right) \\
&= \frac{\left[\frac{d}{dx}(-2x)\right]\left(x^2+1\right)^2 - (-2x)\left[\frac{d}{dx}\left(x^2+1\right)^2\right]}{\left(\left(x^2+1\right)^2\right)^2} \\
&= \frac{[-2]\left(x^2+1\right)^2 + 2x\left[2\left(x^2+1\right)\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^4} \\
&= \frac{-2\left(x^2+1\right)^2 + 4x\left(x^2+1\right)2x}{\left(x^2+1\right)^4} = \frac{-2\left(x^2+1\right) + 8x^2}{\left(x^2+1\right)^3} \\
&= \frac{-2x^2 - 2 + 8x^2}{\left(x^2+1\right)^3} = \frac{6x^2-2}{\left(x^2+1\right)^3} = \frac{2\left(3x^2-1\right)}{\left(x^2+1\right)^3}
\end{aligned}$$

Since $x^2+1>0$ for all x, $\left(x^2+1\right)^3>0$ for all x as well. It follows that $f''(x)$ is defined for all x, and is positive, zero, or negative exactly as $3x^2-1$ is. $3x^2-1$ is >0 when $|x|>\frac{1}{\sqrt{3}}$, is $=0$ when $|x|=\frac{1}{\sqrt{3}}$, *i.e.* $x=\pm\frac{1}{\sqrt{3}}$, and is <0 when $|x|<\frac{1}{\sqrt{3}}$. This means that $y=f(x)$ is concave up when $x<-\frac{1}{\sqrt{3}}$ and when $x>\frac{1}{\sqrt{3}}$, concave down when $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$, and has inflection points when $x=-\frac{1}{\sqrt{3}}$ and when $x=\frac{1}{\sqrt{3}}$. We summarize this in the usual table:

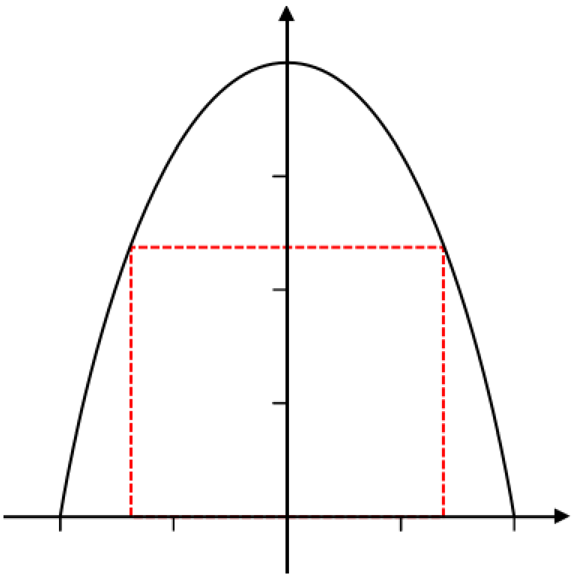
x	$\left(-\infty,-\frac{1}{\sqrt{3}}\right)$	$-\frac{1}{\sqrt{3}}$	$\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$	$\frac{1}{\sqrt{3}}$	$\left(\frac{1}{\sqrt{3}},\infty\right)$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$	infl	$\frown$	infl	$\smile$

vii. Graph. We cheat just a little and have software plot the graph. (In this case, a program called `kmplot`.)



□

b. Here is a sketch of the setup:



It is not hard to see that such a rectangle must have corners with coordinates, going counterclockwise from the lower right, $(x, 0)$, $\left(x, 4 - x^2\right)$, $\left(-x, 4 - x^2\right)$, and $(-x, 0)$, respectively, where $0 \leq x \leq 2$. It follows that such a rectangle would have a base of length $x - (-x) = 2x$ and height of $4 - x^2$, and thus have area $A(x) = 2x\left(4 - x^2\right) = 8x - 2x^3$. Thus we need to find the maximum value of $A(x) = 8x - 2x^3$ for $0 \leq x \leq 2$.

First, we check the endpoints: $A(0) = 8 \cdot 0 - 2 \cdot 0^3 = 0 - 0 = 0$ and $A(2) = 8 \cdot 2 - 2 \cdot 2^3 = 16 - 16 = 0$.

Second we look for critical points between 0 and 2.

$$A'(x) = \frac{d}{dx}\left(8x - 2x^3\right) = 8 \cdot 1 - 2 \cdot 3x^2 = 8 - 6x^2 = 2\left(4 - 3x^2\right)$$

It follows that $A'(x) = 0 \iff 4 - 3x^2 = 0 \iff x^2 = \dfrac{4}{3} \iff x = \pm\dfrac{2}{\sqrt{3}}$. Of the two critical points, we need not consider $x = -\dfrac{2}{\sqrt{3}} < 0$ because it is not between 0 and 2. However, $x = \dfrac{2}{\sqrt{3}} \approx 1.1547$ is between 0 and 2, so we need to check the value of $A(x)$ at this point:

$$A\left(\frac{2}{\sqrt{3}}\right) = 8 \cdot \frac{2}{\sqrt{3}} - 2\left(\frac{2}{\sqrt{3}}\right)^3 = \frac{16}{\sqrt{3}} - \frac{16}{3\sqrt{3}} = \frac{32}{3\sqrt{3}} \approx 6.1584$$

Third, we compare the values of $A(x)$ at the endpoints of the interval of possible x values with the value at the sole critical point in the interval. As $A\left(\dfrac{2}{\sqrt{3}}\right) = \dfrac{32}{3\sqrt{3}} > 0 = A(0) = A(2)$, we see that the maximum possible area for a rectangle with two corners on the x-axis and the other two corners on the part of the parabola $y = 4 - x^2$ that is above the x-axis is $A\left(\dfrac{2}{\sqrt{3}}\right) = \dfrac{32}{3\sqrt{3}} \approx 6.1584$. $\blacksquare$

[Total = 30]