Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2026 (S62)

Quiz #9 – Definitely Integration

Due just before midnight on Wednesday, 22 July.

Evaluate any *four* (4) of the following definite integrals by hand. Please show all the main steps.

1. $\displaystyle\int_0^{\pi/2} 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx$ *[1.25]*

2. $\displaystyle\int_0^1 \frac{2}{\sqrt{x}+x\sqrt{x}}\,dx$ *[1.25]*

3. $\displaystyle\int_2^3 \frac{x^2-5x+4}{x-4}\,dx$ *[1.25]*

4. $\displaystyle\int_0^{1/\sqrt{\ln(41)}} \ln\left(41^x\right)\,dx$ *[1.25]*

5. $\displaystyle\int_0^{\pi/4} x\sec^2(x)\,dx$ *[1.25]*

6. $\displaystyle\int_0^1 \frac{x^3}{e^{x^2}}\,dx$ *[1.25]*

1. $\displaystyle\int_0^{\pi/2} 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx$ *[1]*

SOLUTIONS. **1.** We will use the double angle formula $\sin(2t) = 2\sin(t)\cos(t)$ and the fact that the antiderivative of $\sin(x)$ is $-\cos(x)$, which follows from $\dfrac{d}{dx}\cos(x) = -\sin(x)$.

$$\begin{aligned}\int_0^{\pi/2} 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx &= \int_0^{\pi/2} \sin\left(2\cdot\frac{x}{2}\right)\,dx = \int_0^{\pi/2}\sin(x)\,dx = -\cos(x)\Big|_0^{\pi/2}\\ &= \left[-\cos\left(\frac{\pi}{2}\right)\right] - \left[-\cos(0)\right] = [-0]-[-1] = 1 \quad\square\end{aligned}$$

2. We will use the substitution $w = \sqrt{x}$, so $dw = \dfrac{1}{2\sqrt{x}}\,dx$ and thus $\dfrac{1}{\sqrt{x}}\,dx = 2\,dw$, changing (?) the limits as we go along: $\begin{smallmatrix} x & 0 & 1 \\ w & 0 & 1\end{smallmatrix}$ Note that $x = w^2$.

$$\begin{aligned}\int_0^1 \frac{2}{\sqrt{x}+x\sqrt{x}}\,dx &= \int_0^1 \frac{2}{1+x}\cdot\frac{1}{\sqrt{x}}\,dx = \int_0^1 \frac{2}{1+w^2}\cdot 2\,dw = 4\int_0^1\frac{1}{1+w^2}\,dw\\ &= 4\arctan(w)\Big|_0^1 = 4\arctan(1) - 4\arctan(0)\\ &= 4\cdot\frac{\pi}{4} - 4\cdot 0 = \pi - 0 = \pi \quad\square\end{aligned}$$

3. The key to this one are the observations that $x^2-5x+4 = (x-1)(x-4)$ and that $4 \notin [2,3]$, so for x between 2 and 3 we have $\dfrac{x^2-5x+4}{x-4} = x-1$. Then, with the help of the Power Rule for integration:

$$\begin{aligned}\int_2^3 \frac{x^2-5x+4}{x-4}\,dx &= \int_2^3\frac{(x-4)(x-1)}{x-4}\,dx = \int_2^3 (x-1)\,dx = \left[\frac{x^2}{2}-x\right]_2^3\\ &= \left[\frac{3^2}{2}-3\right] - \left[\frac{2^2}{2}-2\right] = \left[\frac{9}{2}-\frac{6}{2}\right] - [2-2] = \frac{3}{2} - 0 = \frac{3}{2} \quad\square\end{aligned}$$

4. Here we use the property of logarithms that $\ln\left(a^b\right) = b \cdot \ln(a)$ to simplify the integrand, and then apply the Power Rule:

$$\int_0^{1/\sqrt{\ln(41)}} \ln\left(41^x\right)\, dx = \int_0^{1/\sqrt{\ln(41)}} x \cdot \ln(41)\, dx = \left[\ln(41) \cdot \frac{x^2}{2}\right]_0^{1/\sqrt{\ln(41)}}$$

$$= \left[\frac{\ln(41)}{2} \cdot \left(\frac{1}{\sqrt{\ln(41)}}\right)^2\right] - \left[\frac{\ln(41)}{2} \cdot 0^2\right]$$

$$= \left[\frac{\ln(41)}{2} \cdot \frac{1}{\ln(41)}\right] - 0 = \frac{1}{2} - 0 = \frac{1}{2} \quad \square$$

5. We will use integration by parts with $u = x$ and $v' = \sec^2(x)$, so $u' = 1$ and $v = \tan(x)$.

$$\int_0^{\pi/4} x \sec^2(x)\, dx = x\tan(x)\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan(x)\, dx$$

$$= \frac{\pi}{4}\tan\left(\frac{\pi}{4}\right) - 0\tan(0) - \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\, dx$$

Substitute $w = \cos(x)$, so $dw = -\sin(x)\, dx$,
$\sin(x)\, dx = (-1)\, dw$, and $\begin{smallmatrix} x & 0 & \pi/4 \\ w & 1 & 1/\sqrt{2} \end{smallmatrix}$.

$$= \frac{\pi}{4} \cdot 1 - 0 - \int_1^{1/\sqrt{2}} \frac{1}{w}(-1)\, dw = \frac{\pi}{4} + \int_1^{1/\sqrt{2}} w^{-1}\, dw$$

$$= \frac{\pi}{4} + \ln(w)\Big|_1^{1/\sqrt{2}} = \frac{\pi}{4} + \ln(1) - \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{pi}{4} + 0 - \ln\left(2^{-1/2}\right) = \frac{\pi}{4} - \frac{1}{2}\ln(2) \quad \square$$

6. We first substitute $w = x^2$, so $dw = 2x\, dx$ and $x\, dx = \dfrac{1}{2}\, dw$, and change (?) the limits as we go along: $\begin{smallmatrix} x & 0 & 1 \\ w & 0 & 1 \end{smallmatrix}$

$$\int_0^1 \frac{x^3}{e^{x^2}}\, dx = \int_0^1 \frac{x \cdot x^2}{e^{x^2}}\, dx = \int_0^1 \frac{w}{e^w} \cdot \frac{1}{2}\, dw = \frac{1}{2}\int_0^1 we^{-w}\, dw$$

Integration by parts with $u = w$ and $v' = e^{-w}$,
so $u' = 1$ and $v = (-1)e^{-w}$.

$$= \frac{1}{2}\left[w(-1)e^{-w}\Big|_0^1 - \int_0^1 1(-1)e^{-w}\, dw\right]$$

$$= \frac{1}{2}\left[1(-1)e^{-1} - 0(-1)e^{-0} + \int_0^1 e^{-w}\, dw\right]$$

$$= \frac{1}{2}\left[-\frac{1}{e} - 0 + (-1)e^{-w}\Big|_0^1\right] = \frac{1}{2}\left[-\frac{1}{e} + (-1)e^1 - (-1)e^0\right]$$

$$= \frac{1}{2}\left[-\frac{1}{e} - \frac{1}{e} + 1\right] = \frac{1}{2}\left[1 - \frac{2}{e}\right] = \frac{1}{2} - \frac{1}{e} \quad \blacksquare$$