Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2026 (S62)

Solutions to Quiz #2

Due just before midnight on Wednesday, 24 June.

Do any *two* (2) of questions **1**–**3**.

1. Use the ε–δ definition of limits to verify part of the Sum Rule for limits: if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} [f(x) + g(x)] = L + M$. *[2.5 + 0.5 bonus]*

SOLUTION. Suppose that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. By definition, this means that for any $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ such that if $|x - a| < \delta_1$, then $|f(x) - L| < \varepsilon_1$, and also that for any $\varepsilon_2 > 0$, there is a $\delta_2 > 0$ such that if $|x - a| < \delta_2$, then $|f(x) - L| < \varepsilon_2$.

We need to verify that $\lim_{x \to a} [f(x) + g(x)] = L + M$. By definition, this means that we need to check that for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x - a| < \delta$, then $|[f(x) + g(x)] - [L + M]| < \varepsilon$.

Suppose, then, that we are given a generic $\varepsilon > 0$. As usual, we try to find a corresponding $\delta > 0$ by starting with what we're trying to achieve. Observe that

$$\begin{aligned} |[f(x) + g(x)] - [L + M]| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| , \end{aligned}$$

which means we can get $|[f(x) + g(x)] - [L + M]| < \varepsilon$ to happen if we get $|f(x) - L| + |g(x) - M| < \varepsilon$. We can accomplish that by ensuring that $|f(x) - L|$ and $|g(x) - M|$ are less than $\frac{\varepsilon}{2}$ each.

Let $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that if $|x - a| < \delta_1$, then $|f(x) - L| < \varepsilon_1 = \frac{\varepsilon}{2}$, and if $|x - a| < \delta_2$, then $|g(x) - M| < \varepsilon_2 = \frac{\varepsilon}{2}$. Now let $\delta = \min(\delta_1, \delta_2)$. If $|x - a| < \delta$, then $|f(x) - L| < \varepsilon_1 = \frac{\varepsilon}{2}$ because $\delta \leq \delta_1$, and $|g(x) - M| < \varepsilon_2 = \frac{\varepsilon}{2}$ because $\delta \leq \delta_2$. This means that if $|x - a| < \delta$, then

$$|[f(x) + g(x)] - [L + M]| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By the ε–δ definition of limits, it follows that $\lim_{x \to a} [f(x) + g(x)] = L + M$, as desired. ■

2. Use the limit definition of the derivative to verify that $\frac{d}{dx}\left(x^3 + 2x\right) = 3x^2 + 2$. *[2.5]*

Hint: Plug the function into the definition and do algebra …

SOLUTION. Recall that, by definition, $\dfrac{d}{dx}f(x) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$. In this case, $f(x) = x^3+2x$, so here goes:

$$\begin{aligned}
\frac{d}{dx}\left[x^3+2x\right] &= \lim_{h\to 0}\frac{\left[(x+h)^3+2(x+h)\right]-\left[x^3+2x\right]}{h} \\
&= \lim_{h\to 0}\frac{\left[x^3+3x^2h+3xh^2+h^2+2x+2h\right]-\left[x^3+2x\right]}{h} \\
&= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^2+2h}{h} = \lim_{h\to 0}\frac{h\left(3x^2+3xh+h+2\right)}{h} \\
&= \lim_{h\to 0}\left(3x^2+3xh+h+2\right) = 3x^2+3x\cdot 0+0+2 \\
&= 3x+2 \quad \blacksquare
\end{aligned}$$

3. Find the equation of the tangent line to $y=\sqrt{x}$ at $x=1$. *[2.5]*

SOLUTION. We want the equation of the line that passes through the point on the graph of $y=\sqrt{x}$ at $x=1$, namely $\left(1,\sqrt{1}\right)=(1,1)$, that has slope

$$m = \frac{dy}{dx} = \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2}\cdot\frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$$

at $x=1$, namely $m = \dfrac{1}{2\sqrt{1}} = \dfrac{1}{2}$.

The line with slope $\frac{1}{2}$ passing through $(1,1)$ satisfies $\dfrac{y-1}{x-1} = \dfrac{1}{2}$. (Why?) It follows that

$$y-1 = \frac{1}{2}(x-1) \implies y = \frac{1}{2}x - \frac{1}{2} + 1 = \frac{1}{2}x + \frac{1}{2}. \quad \blacksquare$$

[Total = 5]