Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals

Trent University, Summer 2026 (S62)

Solutions to Quiz #11

Due just before midnight on Wednesday, 29 July.

Consider the curve $y = \frac{2}{3}x^{3/2}$, where $0 \le x \le 2$.

1. Find the arc-length of this curve. *[2.5]*

Solution. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2}\right) = \dfrac{2}{3}\cdot\dfrac{3}{2}x^{1/2} = x^{1/2}$. Plugging this into the arc-length integral formula gives:

$$\text{arc-length} = \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{1+\left(x^{1/2}\right)^2}\,dx = \int_0^2 \sqrt{1+x}\,dx$$

Substitute $w = x+1$, so $dw = dx$, and change the limits: $\begin{smallmatrix} x & 0 & 2 \\ w & 1 & 3 \end{smallmatrix}$

$$= \int_1^3 \sqrt{w}\,dw = \int_1^3 w^{1/2}\,dw = \left.\frac{w^{3/2}}{3/2}\right|_1^3 = \left.\frac{2}{3}w^{3/2}\right|_1^3 = \frac{2}{3}\cdot 3^{3/2} - \frac{2}{3}\cdot 1^{3/2}$$

$$= 2\sqrt{3} - \frac{2}{3} \approx 2.7974 \quad \square$$

2. Find the area of the surface obtained by revolving this curve about the y-axis. *[2.5]*

Solution. If we revolve the given curve about the y-axis, the point at x goes a round a circle with radius $r = x - 0 = x$. Plugging this, and $\dfrac{dy}{dx} = x^{1/2}$ into the surface area integral formula gives:

$$\text{area of surface} = \int_0^2 2\pi r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^2 x\sqrt{1+\left(x^{1/2}\right)^2}\,dx$$

$$= 2\pi\int_0^2 x\sqrt{1+x}\,dx \quad \text{Let } w = x+1\text{, so } dw = dx \text{ and } x = w-1,$$

and change the limits: $\begin{smallmatrix} x & 0 & 2 \\ w & 1 & 3 \end{smallmatrix}$

$$= 2\pi\int_1^3 (w-1)w^{1/2}\,dw = 2\pi\int_1^3 \left(w^{3/2} - w^{1/2}\right)dw$$

$$= 2\pi\left.\left(\frac{w^{5/2}}{5/2} - \frac{w^{3/2}}{3/2}\right)\right|_1^3 = 2\pi\left(\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right)\Bigg)_1^3$$

$$= 2\pi\left(\frac{2}{5}\cdot 3^{5/2} - \frac{2}{3}\cdot 3^{3/2}\right) - 2\pi\left(\frac{2}{5}\cdot 1^{5/2} - \frac{2}{3}\cdot 1^{3/2}\right)$$

$$= 2\pi\left(\frac{18\sqrt{3}}{5} - 2\sqrt{3}\right) - 2\pi\left(\frac{2}{5} - \frac{2}{3}\right)$$

$$= 2\pi\left(\frac{18\sqrt{3}}{5} - \frac{10\sqrt{3}}{5}\right) - 2\pi\left(\frac{6}{15} - \frac{10}{15}\right) = 2\pi\left[\frac{8\sqrt{3}}{5} + \frac{4}{15}\right]$$

$$= 2\pi\left[\frac{24\sqrt{3}}{15} + \frac{4}{15}\right] = \frac{8\pi}{15}\left(6\sqrt{3}+1\right) \approx 19.0880 \quad \blacksquare$$