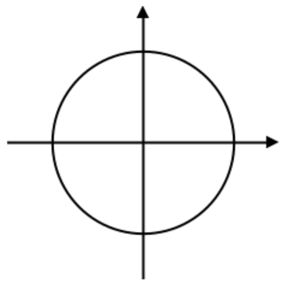
Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2026 (S62) [reading course]

Solutions to Assignment #5
Due on Friday, 24 July.

Consider the region enclosed by the circle $x^2 + y^2 = 4$. You can think of this as the region below the curve $y = \sqrt{4 - x^2}$ and above the curve $y = -\sqrt{4 - x^2}$, where $-2 \le x \le 2$.

1. Sketch or plot this region. What is the area of this region? *[2]*

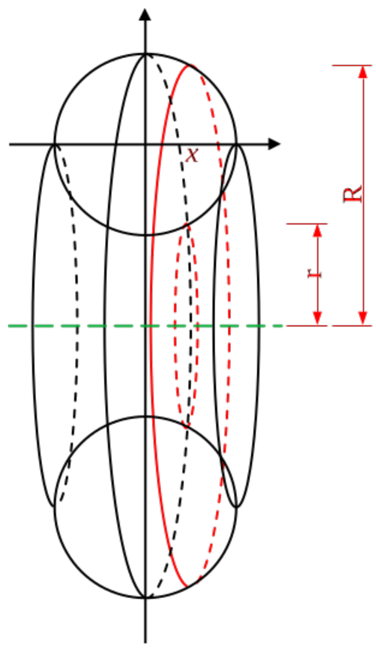
SOLUTION. Using SageMath or the like is too easy, so here's a crude sketch:



The area within a circle of radius 2 – note that the given equation is $x^2 + y^2 = 4 = 2^2$ – is $\pi 2^2 = 4\pi$. ■

2. Sketch the solid obtained by revolving this region about the horizontal line $y = -4$. *[2]*

SOLUTION. Here is a crude sketch:



Note that a "washer" cross-section has been drawn in. ■

3. Use the disk/washer method to compute the volume of the solid in question **2**. *[5]*

NOTE/HINT. You may use SageMath to evaluate the integral, if you're willing to look up the `integral` command, or do it by hand, which would likely mean reinterpreting the integral you get as an area rather than evaluating it directly.

SOLUTION. Since the given region was revolved about a horizontal line, the "washer" cross-sections are parallel to the y-axis and perpendicular to the x-axis. Looking at the sketch given in the solution to question **2**, it is not hard to see that the cross-section at x, where $-2 \le x \le 2$, has outer radius $R = +\sqrt{4-x^2} - (-4) = \sqrt{4-x^2} + 4$ and inner radius $r = -\sqrt{4-x^2} - (-4) = -\sqrt{4-x^2} + 4$, and thus has area

$$\begin{aligned} A(x) &= \pi \left[R^2 - r^2\right] = \pi \left[\left(\sqrt{4-x^2}+4\right)^2 - \left(-\sqrt{4-x^2}+4\right)^2\right] \\ &= \pi \left[\left(4-x^2+8\sqrt{4-x^2}+16\right) - \left(4-x^2-8\sqrt{4-x^2}+16\right)\right] \\ &= 16\pi\sqrt{4-x^2}. \end{aligned}$$

It follows that the volume of the solid is

$$V = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} 16\pi\sqrt{4-x^2}\,dx = 16\pi \int_{-2}^{2} \sqrt{4-x^2}\,dx.$$

We don't really have the tools (trigonometric substitutions) to evaluate this integral by hand. However, $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$ is the area of the upper semi-circle of the given original circle, which is $\dfrac{1}{2} \cdot \pi 2^2 = 2\pi$. Thus the volume of this solid of revolution is

$$V = 16\pi \int_{-2}^{2} \sqrt{4-x^2}\,dx = 16\pi \cdot 2\pi = 32\pi^2. \quad \blacksquare$$

4. There is a general method that would let us compute the volume of the given solid without using calculus or just looking up the formula for the volume of a torus. What is it? *[1]*

HINT. The name to conjure with is "Pappus".

SOLUTION. Pappus' Centroid Theorem – the version concerned with volume – states that the volume $V$of a solid of revolution generated by rotating a plane figure F about an external axis is equal to the product of the area A of F and the distance d traveled by the geometric centroid of F. (The geometric centroid of F is the centre of mass, *i.e.* balance point, of F if the figure has uniform density and thickness.)

It is easy to see that the centroid of a disk is the centre of the disk. In the given case, this is the origin, which is taken around a circle of radius 4 when revolving the region to make the solid, and hence travels through the circumference of $2\pi 4 = 8\pi$. As the area of a disk of radius 2 is $\pi 2^2 = 4\pi$, the volume of the resulting solid is $V = 8\pi \cdot 4\pi = 32\pi^2$. $\blacksquare$