TRENT UNIVERSITY, Summer 2025

MATH 1110H Midterm Test

Monday, 7 July

Time: 60 minutes

Name: Nemo Sum

STUDENT NUMBER: 0000000

Question	Mark	
1	___	
2	___	
3	___	
Total	___	/30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and all sides of one letter- or A4-size aid sheet.
- If you do more than the minimum number of parts or questions, only the first ones the marker finds will be marked. Cross out anything you do not want marked.

1. Do any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. Use the ε–δ definition of limits to check that $\lim_{x \to 3}(4x-5) = 7$.

b. Compute $\displaystyle\lim_{x \to -1} \frac{x^3 - x}{x+1}$.

c. Use the limit definition of the derivative to verify that $\dfrac{d}{dx}x^3 = 3x^2$.

SOLUTIONS. **a.** We need to show that for every $\varepsilon > 0$, we can find a $\delta > 0$ such that if $|x-3| < \delta$, then $|(4x-5)-7| < \varepsilon$.

Given an $\varepsilon > 0$, we reverse-engineer the required $\delta > 0$.

$$\begin{aligned} |(4x-5)-7| < \varepsilon &\iff |4x-12| < \varepsilon \iff |4(x-3)| < \varepsilon \\ &\iff 4|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{4} \end{aligned}$$

Observe that every step above is reversible. If we now set $\delta = \dfrac{\varepsilon}{4}$, it follows that if $|x-3| < \delta = \dfrac{\varepsilon}{4}$, then $|(4x-5)-7| < \varepsilon$ because we can run the chain of reasoning above backwards.

Thus $\lim_{x \to 3}(4x-5) = 7$ by ε–δ definition of limits. □

b. *Using algebra.* We do a little simplification before evaluating the limit because the denominator, $x+1$, approaches 0 as x approaches -1.

$$\begin{aligned} \lim_{x \to -1} \frac{x^3 - x}{x+1} &= \lim_{x \to -1} \frac{x\left(x^2-1\right)}{x+1} = \lim_{x \to -1} \frac{x(x-1)(x+1)}{x+1} \\ &= \lim_{x \to -1} x(x-1) = (-1)(-1-1) = (-1)(-2) = 2 \quad \square \end{aligned}$$

b. *Using l'Hôpital's Rule.* Since both the numerator, $x^3 - x$, and denominator, $x+1$, approach 0 as x approaches -1, we may apply l'Hôpital's Rule.

$$\lim_{x \to -1} \frac{x^3 - x \to 0}{x+1 \;\to 0} = \lim_{x \to -1} \frac{\frac{d}{dx}\left(x^3 - x\right)}{\frac{d}{dx}(x+1)} = \lim_{x \to -1} \frac{3x^2 - 1}{1} = 3(-1)^2 - 1 = 3 - 1 = 2 \quad \square$$

c. We plug $f(x) = x^3$ into the limit definition of the derivative and see what emerges.

$$\begin{aligned} \frac{d}{dx}x^3 &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right) = 3x^2 + 3x \cdot 0 + 0^2 = 3x^2 \end{aligned}$$

Thus $\dfrac{d}{dx}x^3 = 3x^2$ by the limit definition of the derivative. ■

2. Find $\dfrac{dy}{dx}$ in any *two* (2) of parts **a**–**c**. *[10 = 2 × 5 each]*

a. $y = \ln(\sec(x))$ **b.** $y = \dfrac{x+1}{x^3 - x}$ **c.** $e^{x^2+y^2} = 10$

SOLUTIONS. **a.** *Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx} = \ln(\sec(x)) = \frac{1}{\sec(x)} \cdot \frac{d}{dx}\sec(x) = \frac{1}{\sec(x)} \cdot \sec(x)\tan(x) = \tan(x) \quad \square$$

b. *Simplify, then use the Quotient and Power Rules.* $x^3 - x = x\left(x^2 - 1\right) = x(x-1)(x+1)$, so $y = \dfrac{x+1}{x^3-x} = \dfrac{x+1}{x(x-1)(x+1)} = \dfrac{1}{x(x-1)} = \dfrac{1}{x^2-x}$. It follows that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{x^2-x}\right) = \frac{\left[\frac{d}{dx}1\right]\left(x^2-x\right) - 1\left[\frac{d}{dx}\left(x^2-x\right)\right]}{\left(x^2-x\right)^2} \\ &= \frac{0\left(x^2-x\right) - (2x-1)}{\left(x^2-x\right)^2} = \frac{-2x+1}{\left(x^2-x\right)^2} \quad \square\end{aligned}$$

NOTE. Using the Quotient and Power Rules first, and then simplifying, requires more difficult algebra when simplifying.

c. *Solve for y, then differentiate.*

$$\begin{aligned}e^{x^2+y^2} = 10 &\iff x^2 + y^2 = \ln\left(e^{x^2+y^2}\right) = \ln(10) \\ &\iff y^2 = \ln(10) - x^2 \iff y = \pm\sqrt{\ln(10) - x^2}\end{aligned}$$

It follows that

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\pm\sqrt{\ln(10)-x^2}\right) = \pm\frac{1}{2\sqrt{\ln(10)-x^2}} \cdot \frac{d}{dx}\left(\ln(10) - x^2\right) \\ &= \frac{\pm 1}{2\sqrt{\ln(10)-x^2}} \cdot (-2x) = \frac{\mp x}{\sqrt{\ln(10)-x^2}}. \quad \square\end{aligned}$$

c. *Simplify, then use implicit differentiation.*

$$e^{x^2+y^2} = 10 \iff x^2 + y^2 = \ln\left(e^{x^2+y^2}\right) = \ln(10)$$

Differentiate on both sides:

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}\ln 10 \iff 2x + 2y\frac{dy}{dx} = 0 \iff \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y} \quad \blacksquare$$

NOTE. Using implicit differentiation first and then simplifying is a little bit more work, but should give the same answer.

3. Do *one* (1) of parts **a** or **b**. *[10]*

a. Find the domain as well as any and all intercepts, horizontal and vertical asymptotes, intervals of increase and decrease, local maximum and minimum points, intervals of concavity, and inflection points of $f(x) = e^{-x^2}$, and sketch its graph based on this information.

b. Two very long straight walls meet at right angles.
i. A triangular plot is to be created by cutting off this corner with a straight fence from one wall to the other. What is the maximum possible area of the plot if the fence is 25 m long? *[7]*
ii. A quarter-disk plot is to be created by cutting off this corner with a fence that is a circular arc centred at the corner. What is the area of this plot if this fence is 25 m long? *[3]*

SOLUTIONS. **a.** We run through the indicated checklist:

i. Domain. Since $-x^2$ is defined for all x and e^t is defined for all t, $f(x) = e^{-x^2}$ is defined for all x. That is, the domain of $f(x)$ is $\mathbb{R} = (-\infty, \infty)$.

ii. Intercepts. $e^{-0^2} = e^0 = 1$, so $y = f(x)$ has its y-intercept at $y = 1$.
Since $e^t > 0$ for all t, e^{-x^2} is never 0, so $y = f(x)$ has no x-intercepts.

iii. Vertical asympotes. Since $f(x) = e^{-x^2}$ is defined for all x and is continuous wherever it is defined (being a composition of contonuous functions), it does not have any vertical asymptotes.

iv. Horizontal asymptotes. We check the behaviour of $f(x)$ as $x \to -\infty$ and as $x \to \infty$. Note that $-x^2 \to -\infty$ as $x \to -\infty$ and as $x \to \infty$.

$$\lim_{x\to-\infty} e^{-x^2} = \lim_{t\to-\infty} e^t = 0^+$$
$$\lim_{x\to\infty} e^{-x^2} = \lim_{t\to-\infty} e^t = 0^+$$

Thus $y = f(x)$ has $y = 0$ as an asymptote in both directions, which it approaches from above in both directions.

v. Increase/decrease/maxima/minima. We compute the derivative ...

$$f'(x) = \frac{d}{dx}e^{-x^2} = e^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right) = e^{-x^2} \cdot (-2x) = -2xe^{-x^2}$$

... and then use it. Since $e^{-x^2} > 0$ for all x,

$$f'(x) = -2xe^{-x^2} \begin{matrix} <0 \\ =0 \\ >0 \end{matrix} \iff -2x \begin{matrix} <0 \\ =0 \\ >0 \end{matrix} \iff x \begin{matrix} >0 \\ =0 \\ <0 \end{matrix}.$$

This means $f(x)$ is increasing when $x < 0$ and decreasing when $x > 0$, so it has a local maximum at $x = 0$. We summarize this in a table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	$\uparrow$	max	$\downarrow$

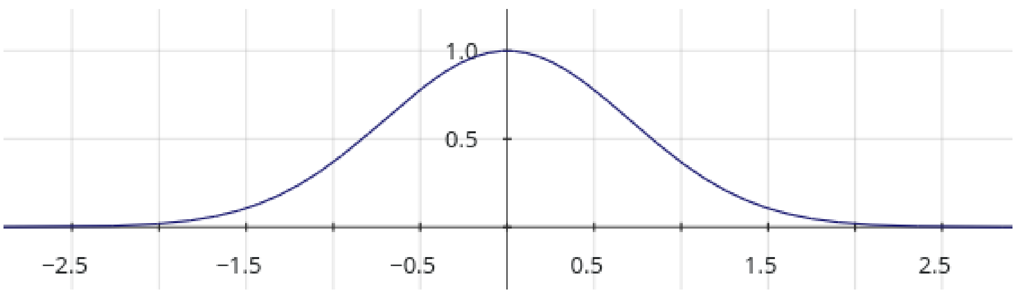
vi. Intervals of concavity and inflection points. We compute the second derivative ...

$$\begin{aligned} f''(x) &= \frac{d}{dx}f'(x) = \frac{d}{dx}\left(-2xe^{-x^2}\right) = \left[\frac{d}{dx}(-2x)\right]e^{-x^2} + (-2x)\left[\frac{d}{dx}e^{-x^2}\right] \\ &= [-2]e^{-x^2} + (-2x)e^{-x^2}\left[\frac{d}{dx}\left(-x^2\right)\right] = -2e^{-x^2} + (-2x)e^{-x^2}[-2x] \\ &= \left(4x^2-2\right)e^{-x^2} = 4\left(x^2-\frac{1}{2}\right)e^{-x^2} = 4\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)e^{-x^2} \end{aligned}$$

... and put it to work. Since $4e^{-x^2} > 0$ for all x, $f''(x) = 0$ exactly when $x = \pm\frac{1}{\sqrt{2}}$; $f''(x) < 0$ when exactly one of $x - \frac{1}{\sqrt{2}}$ or $x + \frac{1}{\sqrt{2}}$ is < 0, *i.e.* when $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$; and $f''(x) > 0$ when $x - \frac{1}{\sqrt{2}}$ and $x + \frac{1}{\sqrt{2}}$ are both > 0 or both < 0, *i.e.* when $x < -\frac{1}{\sqrt{2}}$ and when $x > \frac{1}{\sqrt{2}}$. This means that $f(x)$ is concave up when $x < -\frac{1}{\sqrt{2}}$ and when $x > \frac{1}{\sqrt{2}}$, concave down when $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, and has inflection points when $x = \pm\frac{1}{\sqrt{2}}$. We summarize this in a table:

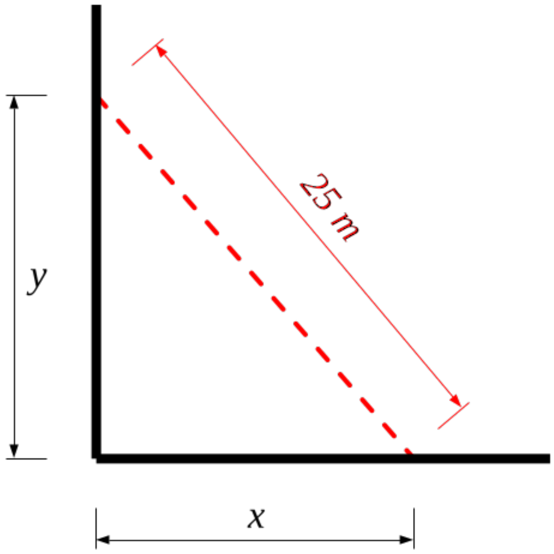
x	$\left(-\infty, \frac{-1}{\sqrt{2}}\right)$	$\frac{-1}{\sqrt{2}}$	$\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, \infty\right)$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$	infl	$\frown$	infl	$\smile$

vii. Graph. We cheat and let a computer program called KmPlot do the job:



■

b. *i.* Here's a picture of the setup:



The two walls and the fence form a right triangle with base x and height y (along the walls), with the 25 m fence forming the hypotenuse . Note that we must have $0 \leq x \leq 25$ and $0 \leq y \leq 25$, and that $x = 0$ exactly when $y = 25$ and $y = 0$ exactly when $x = 25$. This triangle has area $A = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{xy}{2}$ and, by the Pythagorean Theorem, we have $x^2 + y^2 = 25^2 = 625$.

We first solve for y in terms of x from the Pythagorean relation,

$$x^2 + y^2 = 25^2 \iff y^2 = 25^2 - x^2 \iff y = \pm\sqrt{25^2 - x^2}.$$

Since $y \geq 0$, we have $y = \sqrt{25^2 - x^2}$. Plugging this into the area formula for the triangle now gives the area in terms of x only: $A(x) = \frac{xy}{2} = \frac{x\sqrt{25^2 - x^2}}{2}$, where $0 \leq x \leq 25$.

To find possible maxima we differentiate and look for critical points.

$$\begin{aligned}
A'(x) &= \frac{d}{dx}\left(\frac{x\sqrt{25^2 - x^2}}{2}\right) = \frac{1}{2}\left(\left[\frac{d}{dx}x\right]\sqrt{25^2 - x^2} + x\left[\frac{d}{dx}\sqrt{25^2 - x^2}\right]\right) \\
&= \frac{1}{2}\left([1]\sqrt{25^2 - x^2} + x\left[\frac{1}{2\sqrt{25^2 - x^2}} \cdot \frac{d}{dx}\left(25^2 - x^2\right)\right]\right) \\
&= \frac{1}{2}\left(\sqrt{25^2 - x^2} + x \cdot \frac{1}{2\sqrt{25^2 - x^2}} \cdot (-2x)\right) = \frac{1}{2}\left(\sqrt{25^2 - x^2} - \frac{x^2}{\sqrt{25^2 - x^2}}\right)
\end{aligned}$$

Note that

$$\begin{aligned}
A'(x) = 0 &\iff \sqrt{25^2 - x^2} - \frac{x^2}{\sqrt{25^2 - x^2}} = 0 \iff \left(25^2 - x^2\right) - x^2 = 0 \\
&\iff \frac{25^2 - 2x^2}{2} = 0 \iff 2x^2 = 25^2 \iff x^2 = \frac{25^2}{2} \iff x = \pm\frac{25}{\sqrt{2}}.
\end{aligned}$$

Since $0 \le x \le 25$, we need only consider $x = \dfrac{25}{\sqrt{2}}$. Observe that when $x < \dfrac{25}{\sqrt{2}}$, we get

$$25^2 - 2x^2 > 0 \Leftrightarrow \left(25^2 - x^2\right) - x^2 > 0 \Leftrightarrow A'(x) = \frac{1}{2}\left(\sqrt{25^2 - x^2} - \frac{x^2}{\sqrt{25^2 - x^2}}\right) > 0,$$

and when $x > \dfrac{25}{\sqrt{2}}$, we similarly get $A'(x) = \dfrac{1}{2}\left(\sqrt{25^2 - x^2} - \dfrac{x^2}{\sqrt{25^2 - x^2}}\right) < 0$. Thus $A(x)$ increases for $x < \dfrac{25}{\sqrt{2}}$ and decreases for $x > \dfrac{25}{\sqrt{2}}$, which makes $x = \dfrac{25}{\sqrt{2}}$ a local maximum. This maximum is

$$A\left(\frac{25}{\sqrt{2}}\right) = \frac{\frac{25}{\sqrt{2}}\sqrt{25^2 - \left(\frac{25}{\sqrt{2}}\right)^2}}{2} = \frac{\frac{25}{\sqrt{2}}\sqrt{25^2 - \frac{25^2}{2}}}{2}$$

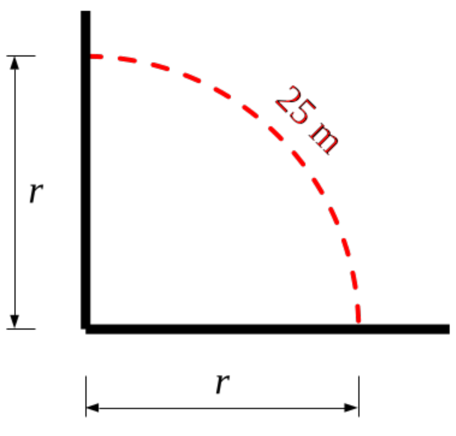
$$= \frac{\frac{25}{\sqrt{2}}\sqrt{\frac{25^2}{2}}}{2} = \frac{\frac{25}{\sqrt{2}} \cdot \frac{25}{\sqrt{2}}}{2} = \frac{25^2}{4} = \frac{625}{4} = 156.25\,.$$

For the sharp-eyed, the fact that we did not consider $x = 25$, where $A'(x)$ is undefined, as a possible critical point, is made up for by the fact that we do consider it when checking the endpoints. At the endpoints, we get zero area:

$$A(0) = \frac{0\sqrt{25^2 - 0^2}}{2} = \frac{0 \cdot 25}{2} = 0 \qquad A(25) = \frac{25\sqrt{25^2 - 25^2}}{2} = \frac{25 \cdot 0}{0} = 0$$

Thus the maximum area of such a plot is $A\left(\dfrac{25}{\sqrt{2}}\right) = \dfrac{25^2}{4} = 156.25\ m^2$. □

b. *ii.* Here is a picture of the setup:



The circumference of a circle with radius r is $2\pi r$. A quarter of that is $\dfrac{\pi r}{2}$, which in this case is equal to $25\ m$. Solving $\dfrac{\pi r}{2} = 25$ for r tells us that $r = \dfrac{50}{\pi}$. The area of a circle of this radius is $\pi r^2 = \pi\left(\dfrac{50}{\pi}\right)^2 = \dfrac{2500}{\pi}$, one quarter of which is $\dfrac{2500}{4\pi} = \dfrac{625}{\pi} \approx 198.9437$. Thus this is the area of the quarter-circular plot. Observe that we didn't use any calculus … ■

[Total = 30]