Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2025 (S62)

Solution to Quiz #8 – Integration
Due on Thursday, 17 July.

1. Compute the indefinite integral $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$ by hand, showing (at least) all the major steps. *[5]*

SOLUTION. Our main tool will be substitution. There are several plausible substitutions one can try, such as $u = e^x$ and $w = e^x + 1$ (both of which work), but we will go whole hog and use the substitution $z = \sqrt{e^x+1}$. Then

$$\frac{dz}{dx} = \frac{d}{dx}\sqrt{e^x+1} = \frac{1}{2\sqrt{e^x+1}} \cdot \frac{d}{dx}\left(e^x+1\right) = \frac{1}{2\sqrt{e^x+1}} \cdot e^x = \frac{e^x}{2\sqrt{e^x+1}},$$

so $dz = \dfrac{e^x}{2\sqrt{e^x+1}}\,dx$. Note that $\dfrac{e^x}{2\sqrt{e^x+1}}$ is very close to the integrand we're dealing with, namely $\dfrac{e^{2x}}{\sqrt{e^x+1}}$. How close? Using the fact that $e^{2x} = e^x e^x$, this close:

$$\frac{e^{2x}}{\sqrt{e^x+1}} = \frac{2e^x e^x}{2\sqrt{e^x+1}} = 2e^x \cdot \frac{e^x}{2\sqrt{e^x+1}}$$

Thus $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \int 2e^x\,dz$. We need to express e^x in terms of z to integrate the latter, but that isn't too hard:

$$z = \sqrt{e^x+1} \iff z^2 = e^x + 1 \iff e^x = z^2 - 1$$

It follows that

$$\begin{aligned}
\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx &= \int 2e^x\,dz = \int 2\left(z^2-1\right)\,dz = 2\int\left(z^2-1\right)\,dz = 2\left(\frac{z^3}{3} - z\right) + C \\
&= 2\left(\frac{\left(\sqrt{e^x+1}\right)^3}{3} - \sqrt{e^x+1}\right) + C = \frac{2}{3}\left(e^x+1\right)^{3/2} - 2\left(e^x+1\right)^{1/2} + C, \\
\text{or, say,} \quad &= 2\sqrt{e^x+1}\left(\frac{e^x+1}{3} - 1\right) + C = 2\sqrt{e^x+1}\left(\frac{e^x}{3} - \frac{2}{3}\right) + C \\
&= \frac{2}{3}\left(e^x-2\right)\sqrt{e^x+1} + C,
\end{aligned}$$

among other forms one could put the answer into ... ■