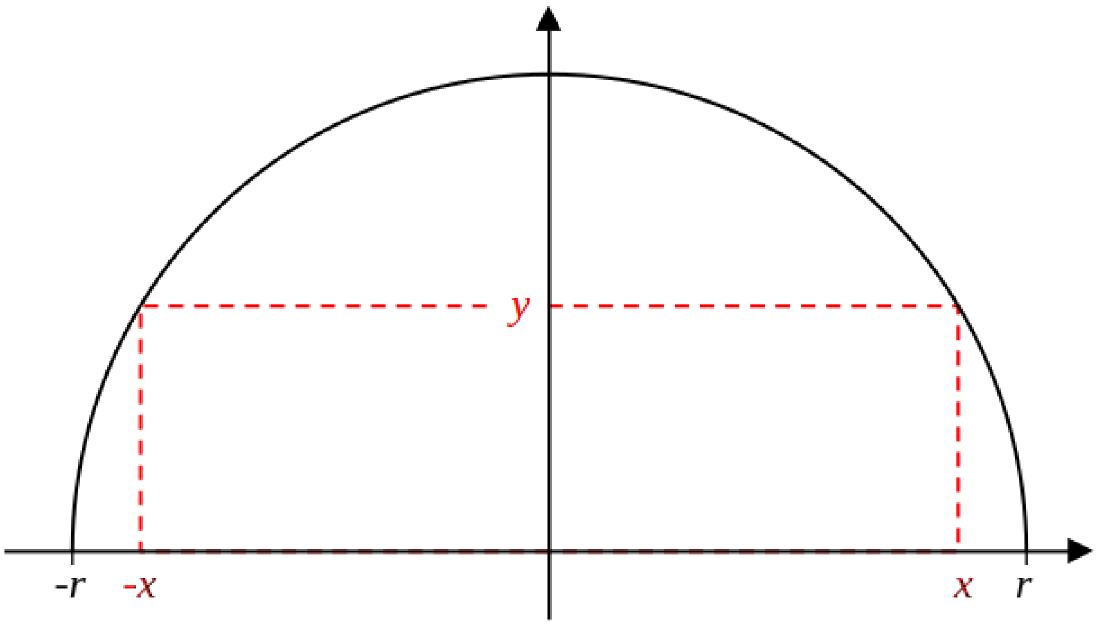
Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2025 (S62)

Quiz #5 – Optimization
Due on Tuesday, 8 July.

1. Find the maximum possible area of a rectangle that can be inscribed in a semicircle of radius r, as in the diagram below. *[5]*



SOLUTION. *i. Diagram.* See above! Annotated a bit from what was originally provided.

ii. Variables, relations, and formulas. Suppose the corners of the inscribed rectangle, going counterclockwise from the bottom right corner, are at $(x, 0)$, (x, y), $(-x, y)$, and $(-x, 0)$, as in the diagram above. Then $x^2 + y^2 = r^2$ because (x, y) is on the circle of radius r centred at the origin, and the area of the rectangle is $A = \text{width} \cdot \text{height} = (x - (x))\,(y - 0) = 2xy$.

If we choose to use x as the fundamental variable, then $0 \le x \le r$ and $y = \sqrt{r^2 - x^2}$, so the area of the rectangle is given by $A = 2x\sqrt{r^2 - x^2}$. This area is what we wish to maximize.

iii. Local maximum. We first look for any critical points $A = 2x\sqrt{r^2 - x^2}$ may have between 0 and r.

$$\begin{aligned}
\frac{dA}{dx} &= \frac{d}{dx} 2x\sqrt{r^2 - x^2} = \left[\frac{d}{dx} 2x\right] \sqrt{r^2 - x^2} + 2x \left[\frac{d}{dx} \sqrt{r^2 - x^2}\right] \\
&= 2\sqrt{r^2 - x^2} + 2x \cdot \frac{1}{2\sqrt{r^2 - x^2}} \cdot \frac{d}{dx}\left(r^2 - x^2\right) \\
&= 2\sqrt{r^2 - x^2} + 2x \cdot \frac{1}{2\sqrt{r^2 - x^2}} \cdot (-2x) = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}}
\end{aligned}$$

It follows that, multiplying through by $\sqrt{r^2 - x^2}$ in $\dfrac{dA}{dx}$,

$$\frac{dA}{dx} = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = 0 \iff 2\left(r^2 - x^2\right) - 2x^2 = 0 \iff 2r^2 - 4x^2 = 0$$
$$\iff x^2 = \frac{2r^2}{4} \iff x = \pm\frac{r}{\sqrt{2}}$$

Of the two possibilities, only the critical point $x = +\dfrac{r}{\sqrt{2}}$ is in the interval $[0, r]$. We check to see if it is a local maximum. Note that $\sqrt{r^2 - x^2} \geq 0$ whenever it is defined.

$$\frac{dA}{dx} = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} < 0 \iff 2\left(r^2 - x^2\right) - 2x^2 < 0 \iff 2r^2 - 4x^2 < 0$$
$$\iff \frac{2r^2}{4} < x^2 \iff \frac{r}{\sqrt{2}} < x$$
$$\frac{dA}{dx} = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} > 0 \iff 2\left(r^2 - x^2\right) - 2x^2 > 0 \iff 2r^2 - 4x^2 > 0$$
$$\iff \frac{2r^2}{4} > x^2 \iff \frac{r}{\sqrt{2}} > x$$

It follows that $A = 2x\sqrt{r^2 - x^2}$ is increasing for $0 < x < \dfrac{r}{\sqrt{2}}$ and decreasing for $\dfrac{r}{\sqrt{2}} < x < r$. Thus $x = \dfrac{r}{\sqrt{2}}$ gives a local maximum for A, and

$$A\left(\frac{r}{\sqrt{2}}\right) = 2 \cdot \frac{r}{\sqrt{2}} \cdot \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} = 2 \cdot \frac{r}{\sqrt{2}} \cdot \sqrt{\frac{r^2}{2}} = 2 \cdot \frac{r}{\sqrt{2}} \cdot \frac{r}{\sqrt{2}} = r^2$$

Note that at the endpoint $x = r$, $\dfrac{dA}{dx}$ is undefined, and so should be considered as a possible critical point as well, but it will be considered with the other endpoint soon.

iv. Endpoints. As noted above we need to consider $A(x) = 2x\sqrt{r^2 - x^2}$ at the endoints of the domain $0 \leq x \leq r$.

$$A(0) = 2 \cdot 0 \cdot \sqrt{r^2 - 0^2} = 2 \cdot 0 \cdot r = 0$$
$$A(r) = 2 \cdot r \cdot \sqrt{r^2 - r^2} = 2 \cdot r \cdot 0 = 0$$

Thus $A(x)$ is smaller at the endpoints than at the critical point $x = \dfrac{r}{\sqrt{2}}$, so its value at the critical point is an absolute maximum.

v. Conclusion. The maximum area of a rectangle inscribed in a semicircle of radius r is $A\left(\dfrac{r}{\sqrt{2}}\right) = r^2$. ■