Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2025 (S62)

Quiz #4 – Even More Derivatives
Due on Thursday, 3 July.

1. Work out $\frac{dy}{dx}$ by hand if $\ln(xy) = 0$, showing all the steps and simplifying your answer as much as you can. *[1]*

SOLUTION 1. *Solve for y first.* $\ln(xy) = 0 \iff xy = 1 \iff y = \frac{1}{x} = x^{-1}$ It follows that $\frac{dy}{dx} = \frac{d}{dx}x^{-1} = (-1)x^{-2} = -\frac{1}{x^2}$. $\square$

SOLUTION 2. *Use implicit differentiation.*

$$\begin{aligned}
\ln(xy) = 0 &\iff \frac{d}{dx}\ln(xy) = \frac{d}{dx}0 \iff \frac{1}{xy} \cdot \frac{d}{dx}(xy) = 0 \\
&\iff \frac{d}{dx}(xy) = 0 \quad \text{since a fraction with numerator 1 can't be 0} \\
&\iff \left[\frac{d}{dx}x\right] y + x \left[\frac{d}{dx}y\right] = 0 \\
&\iff 1 \cdot y + x \cdot \frac{dy}{dx} = 0 \iff \frac{dy}{dx} = -\frac{y}{x} \quad \blacksquare
\end{aligned}$$

2. Find the domain, intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, (local) maximum and minimum points, intervals of curvature, and inflection points of $y = x^3 - x$, and sketch the graph based on this information. *[4]*

SOLUTION. We run through the indicated checklist:

i. Domain. $y = x^3 - x$ is defined for all x, so the domain of the function is $\mathbb{R} = (-\infty, \infty) =$ "all x."

ii. Intercepts. When $x = 0$, $y = 0^3 - 0 = 0$, so the y-intercept is 0.

To get $y = 0$, we need to have $0 = y = x^3 - x = x\left(x^2 - 1\right) = x(x-1)(x+1)$. It follows that $y = 0$ when $x = 0$, $x = 1$, and $x = -1$, *i.e.* the x-intercepts are, from left to right, -1, 0, and 1.

iii. Vertical asymptotes. Since $y = x^3 - x$ is defined and continuous for all x, it does not have any vertical asymptotes.

iv. Horizontal asymptotes. We check in both directions, a little informally.

$$\begin{aligned}
\lim_{x\to-\infty}\left(x^3 - x\right) &= \lim_{x\to-\infty} x\left(x^2 - 1\right) = -\infty \cdot \infty = -\infty \\
\lim_{x\to\infty}\left(x^3 - x\right) &= \lim_{x\to\infty} x\left(x^2 - 1\right) = \infty \cdot \infty = \infty
\end{aligned}$$

Since neither limit has a real number as a limit, $y = x^3 - x$ has no horizontal asymptotes.

v. Increase/decrease/max/min. We compute and then analyze $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - x\right) = 3x^2 - 1$$

This equals 0 when $3x^2 = 1$, *i.e.* when $x = \pm\frac{1}{\sqrt{3}}$; it is less than 0 when$3x^2 < 1$, *i.e.* when $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$; and it is greater than 0 when $3x^2 > 1$, *i.e.* when $x < -\frac{1}{\sqrt{3}}$ and when $x > \frac{1}{\sqrt{3}}$. It follows that $y = x^3 - x$ is increasing when $x < \frac{1}{\sqrt{3}}$, decreasing when $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, and increasing again when $x > \frac{1}{\sqrt{3}}$, and so there is a local maximum at $x = -\frac{1}{\sqrt{3}}$ and a local minimum at $x = \frac{1}{\sqrt{3}}$.

We summarize all this in a table:

x	$\left(-\infty, -\frac{1}{\sqrt{3}}\right)$	$-\frac{1}{\sqrt{3}}$	$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{\sqrt{3}}$	$\left(\frac{1}{\sqrt{3}}, \infty\right)$
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$
y	$\uparrow$	max	$\downarrow$	min	$\uparrow$

Note that $y = x^3 - x$ has no absolute maximum or minimum. There being no horizontal asymptote, it increases without any bound as $x \to \infty$ and decreases without any bound as $x \to -\infty$.

vi. Curvature and inflection. We compute and then analyze $\frac{d^2y}{dx^2}$.

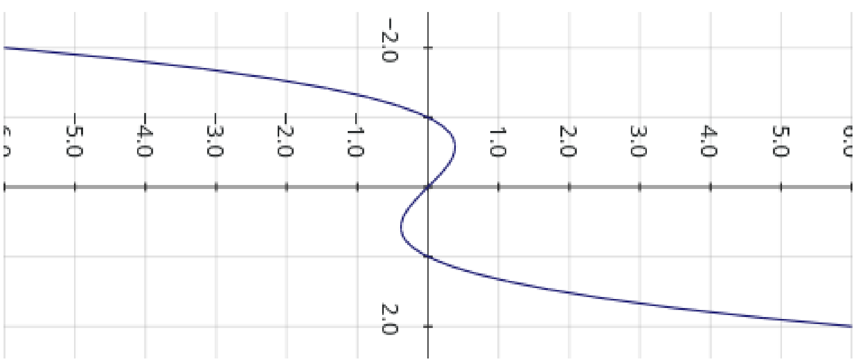
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(3x^2 - 1\right) = 6x$$

This is equal to, less than, or greater that 0 exactly when x is. Thus $y = x^3 - x$ is concave down when $x < 0$ and concave up when $x > 0$, and so there is an inflection point at $x = 0$

We summarize this in another table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$\frac{d^2y}{dx^2}$	$-$	0	$+$
y	$\frown$	infl	$\smile$

vii. Graph. Being lazy, we cheat a bit and have a computer draw the picture.



Instead of SageMath, this was drawn by a dedicated graphing program called KmPlot. ■