Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2025 (S62)

Quiz #3 – More Derivatives
Due on Thursday, 26 June.

1. Work out $\frac{d}{dx}\arccos(x)$ by hand, showing all the steps and simplifying your answer as much as you can. *[2.5]*

SOLUTION. arccos is the inverse function of cos, so $x = \cos(\arccos(x))$. We differentiate both sides of this equation, using the Chain Rule and the fact that $\frac{d}{dt}\cos(t) = -\sin(t)$:

$$1 = \frac{d}{dx}x = \frac{d}{dx}\cos(\arccos(x)) = -\sin(\arccos(x)) \cdot \frac{d}{dx}\arccos(x)$$

Solving for $\frac{d}{dx}\arccos(x)$ and exploiting the fact that $\sin^2(x) + \cos^2(x) = 1$, so $\sin(x) = \sqrt{1-\cos^2(x)}$, gives us

$$\frac{d}{dx}\arccos(x) = \frac{1}{-\sin(\arccos(x))} = \frac{-1}{\sqrt{1-\cos^2(\arccos(x))}} = \frac{-1}{\sqrt{1-x^2}},$$

also exploiting $\cos(\arccos(x)) = x$ one more time. ■

2. Work out $g'(x)$ by hand, where $g(x) = \ln\left(\frac{1+\sin(x)}{\cos(x)}\right)$, showing all the steps and simplifying your answer as much as you can. *[2.5]*

NOTE. Starting with the three (!) solutions below we shall forego describing exactly which facts/formulas/techniques are being used at each step.

SOLUTION 1. *Exploiting the logarithm first.*

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\ln\left(\frac{1+\sin(x)}{\cos(x)}\right) = \frac{d}{dx}\left[\ln(1+\sin(x)) - \ln(\cos(x))\right] \\
&= \left[\frac{d}{dx}\ln(1+\sin(x))\right] - \left[\frac{d}{dx}\ln(\cos(x))\right] \\
&= \left[\frac{1}{1+\sin(x)} \cdot \frac{d}{dx}(1+\sin(x))\right] - \left[\frac{1}{\cos(x)} \cdot \frac{d}{dx}\cos(x)\right] \\
&= \left[\frac{1}{1+\sin(x)} \cdot (0+\cos(x))\right] - \left[\frac{1}{\cos(x)} \cdot (-\sin(x))\right] \\
&= \frac{\cos(x)}{1+\sin(x)} + \frac{\sin(x)}{\cos(x)} = \frac{\cos(x)\cos(x) + \sin(x)(1+\sin(x))}{\cos(x)(1+\sin(x))} \\
&= \frac{\cos^2(x) + \sin(x) + \sin^2(x)}{\cos(x)(1+\sin(x))} = \frac{1+\sin(x)}{\cos(x)(1+\sin(x))} = \frac{1}{\cos(x)} = \sec(x) \quad \square
\end{aligned}$$

Solution 2. *Rewriting the trigonometry first.*

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}\ln\left(\frac{1+\sin(x)}{\cos(x)}\right) = \frac{d}{dx}\ln\left(\frac{1}{\cos(x)}+\frac{\sin(x)}{\cos(x)}\right) = \frac{d}{dx}\ln\left(\sec(x)+\tan(x)\right) \\
&= \frac{1}{\sec(x)+\tan(x)}\cdot\frac{d}{dx}\left(\sec(x)+\tan(x)\right) \\
&= \frac{1}{\sec(x)+\tan(x)}\cdot\big(\sec(x)\tan(x)+\sec^2(x)\big) \\
&= \frac{\sec(x)\left(\tan(x)+\sec(x)\right)}{\sec(x)+\tan(x)} = \sec(x) \quad \square
\end{aligned}
$$

Solution 3. *Brutal force.* We differentiate right away ...

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}\ln\left(\frac{1+\sin(x)}{\cos(x)}\right) = \frac{1}{\frac{1+\sin(x)}{\cos(x)}}\cdot\frac{d}{dx}\left(\frac{1+\sin(x)}{\cos(x)}\right) \\
&= \frac{\cos(x)}{1+\sin(x)}\cdot\frac{\left[\frac{d}{dx}\left(1+\sin(x)\right)\right]\cos(x)-\left(1+\sin(x)\right)\left[\frac{d}{dx}\cos(x)\right]}{\cos^2(x)} \\
&= \frac{\cos(x)}{1+\sin(x)}\cdot\frac{\left[0+\cos(x)\right]\cos(x)-\left(1+\sin(x)\right)\left[-\sin(x)\right]}{\cos^2(x)} \\
&= \frac{\cos(x)}{1+\sin(x)}\cdot\frac{\cos^2(x)+\sin(x)+\sin^2(x)}{\cos^2(x)} \\
&= \frac{\cos(x)}{1+\sin(x)}\cdot\frac{1+\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)} = \sec(x) \quad \blacksquare
\end{aligned}
$$