Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2025 (S62)

Solutions to Quiz #2 – Derivatives
Due on Tuesday, 24 June.

1. Let $f(x) = x\ln\left(x^2\right)$. Compute $f'(x)$ by hand using the practical rules for computing derivatives and knowledge of the derivatives of common functions. Show all the steps! *[1.5]*

SOLUTION. We will use the Product, Chain, and Power Rules, as well as the fact that $\frac{d}{dt}\ln(t) = \frac{1}{t}$, and some algebra.

$$\begin{aligned} f'(x) &= \frac{d}{dx}f(x) = \frac{d}{dx}\left[x\ln\left(x^2\right)\right] = \left[\frac{d}{dx}x\right]\cdot\ln\left(x^2\right) + x\cdot\left[\frac{d}{dx}\ln\left(x^2\right)\right] \quad \text{[Product Rule]} \\ &= [1]\cdot\ln\left(x^2\right) + x\cdot\left[\frac{1}{x^2}\cdot\frac{d}{dx}\left(x^2\right)\right] \quad \text{[Power \& Chain Rules, } \tfrac{d}{dt}\ln(t) = \tfrac{1}{t}\text{]} \\ &= \ln\left(x^2\right) + x\cdot\frac{1}{x^2}\cdot 2x \quad \text{[Power Rule]} = \ln\left(x^2\right) + \frac{2x^2}{x^2} = \ln\left(x^2\right) + 2 \end{aligned}$$

Those so inclined may use the properties of logarithms to rewrite $\ln\left(x^2\right)$ as $2\ln(x)$, either at the start of the calculation or at the end. □

2. Let $g(x) = \dfrac{\cos\left(e^x\right)}{e^x}$. Compute $g'(x)$ by hand using the practical rules for computing derivatives and knowledge of the derivatives of common functions. Show all the steps! *[1.5]*

SOLUTION. We will use the Quotient and Chain Rules, as well as the facts that $\frac{d}{dt}\cos(t) = -\sin(t)$ and $\frac{d}{dt}e^t = e^t$, and some algebra.

$$\begin{aligned} g'(x) &= \frac{d}{dx}g(x) = \frac{d}{dx}\left[\frac{\cos\left(e^x\right)}{e^x}\right] = \frac{\left[\frac{d}{dx}\cos\left(e^x\right)\right]\cdot e^x - \cos\left(e^x\right)\cdot\left[\frac{d}{dx}e^x\right]}{\left(e^x\right)^2} \quad \text{[Quotient Rule]} \\ &= \frac{\left[-\sin\left(e^x\right)\cdot\frac{d}{dx}\left(e^x\right)\right]e^x - \cos\left(e^x\right)\left[e^x\right]}{\left(e^x\right)^2} \quad \text{[Chain Rule, } \tfrac{d}{dt}\cos(t) = -\sin(t),\ \tfrac{d}{dt}e^t = e^t\text{]} \\ &= \frac{\left[-\sin\left(e^x\right)\cdot e^x\right]e^x - \cos\left(e^x\right)\cdot e^x}{\left(e^x\right)^2} \quad \left[\tfrac{d}{dt}e^t = e^t\right] = \frac{-\sin\left(e^x\right)\cdot\left(e^x\right)^2 - \cos\left(e^x\right)\cdot e^x}{\left(e^x\right)^2} \\ &= -\sin\left(e^x\right) - \frac{\cos\left(e^x\right)}{e^x} \qquad \square \end{aligned}$$

3. Given that we know that $\frac{d}{dx}\sin(x) = \cos(x)$ for all x and that $\sin(0) = 0$ and $\cos(0) = 1$, what does this tell us about what the value of $\lim_{h\to 0}\frac{\sin(h)}{h}$ has to be? Explain! *[2]*

Hint: Look at the definition of $\frac{d}{dx}\sin(x)$ when $x = 0$.

SOLUTION. On the one hand, given what we know, we have

$$\sin'(0) = \left.\frac{d}{dx}\sin(x)\right|_{x=0} = \cos(x)|_{x=0} = \cos(0) = 1.$$

On the other hand, by the limit definition of the derivative,

$$\sin'(0) = \left.\frac{d}{dx}\sin(x)\right|_{x=0} = \lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h} = \lim_{h\to 0}\frac{\sin(h)-0}{h} = \lim_{h\to 0}\frac{\sin(h)}{h}.$$

It follows that $\lim_{h\to 0}\frac{\sin(h)}{h} = \sin'(0) = 1$. ■