Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2025 (S62)

Solutions to the Final Examination
19:00-22:00 in ENW 117 on Tuesday, 29 July.

Instructions: Do both of parts **I** and **II**, and, if you wish, part **III**. Please show all your work, justify all your answers, and simplify these where you reasonably can. When you are asked to do k of n questions, only the first k that are not crossed out will be marked. *If you have a question, or are in doubt about something,* **ask!**

Aids: Any calculator, as long as it can't communicate with other devices; (all sides of) one letter- or A4-size sheet; one organic brain belonging to you.

Part I. Do all four (4) of **1**–**4**.

1. Compute $\frac{dy}{dx}$ as best you can in any four (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. $y = \frac{9-x^2}{3+x}$ **b.** $y = \frac{\cos(x)}{1+\sin(x)}$ **c.** $y = \frac{x}{\ln(x)}$

d. $y = (e^x+3)^5$ **e.** $y = x\tan\left(x^2\right)$ **f.** $y = x^2e^x$

SOLUTIONS. **a.** *Algebra and the Power Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{9-x^2}{3+x}\right) = \frac{d}{dx}\left(\frac{(3-x)(3+x)}{3+x}\right) = \frac{d}{dx}(3-x) = -1 \quad \square$$

a. *Quotient and Power Rules.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{9-x^2}{3+x}\right) = \frac{\left[\frac{d}{dx}\left(9-x^2\right)\right](3+x) - \left(9-x^2\right)\left[\frac{d}{dx}(3+x)\right]}{(3+x)^2} \\ &= \frac{[-2x](3+x) - \left(9-x^2\right)[1]}{(3+x)^2} = \frac{-6x-2x^2-9+x^2}{(3+x)^2} = \frac{-x^2-6x-9}{(3+x)^2} \\ &= -\frac{(3+x)^2}{(3+x)^2} = -1 \quad \square\end{aligned}$$

b. *Quotient Rule and a little trigonometry.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\cos(x)}{1+\sin(x)}\right) = \frac{\left[\frac{d}{dx}\cos(x)\right](1+\sin(x)) - \cos(x)\left[\frac{d}{dx}(1+\sin(x))\right]}{(1+\sin(x))^2} \\ &= \frac{[-\sin(x)](1+\sin(x)) - \cos(x)[\cos(x)]}{(1+\sin(x))^2} = \frac{-\sin(x)-\sin^2(x)-\cos^2(x)}{(1+\sin(x))^2} \\ &= -\frac{\sin(x)+\sin^2(x)+\cos^2(x)}{(1+\sin(x))^2} = -\frac{\sin(x)+1}{(1+\sin(x))^2} = -\frac{1}{1+\sin(x)} \quad \square\end{aligned}$$

c. *Quotient Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{\ln(x)}\right) = \frac{\left[\frac{d}{dx}x\right]\ln(x) - x\left[\frac{d}{dx}\ln(x)\right]}{(\ln(x))^2} = \frac{[1]\ln(x) - x\left[\frac{1}{x}\right]}{(\ln(x))^2} = \frac{\ln(x)-1}{(\ln(x))^2} \quad \square$$

d. *Power and Chain Rules.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x+3\right)^5 = 5\left(e^x+3\right)^4 \cdot \frac{d}{dx}\left(e^x+3\right) = 5\left(e^x+3\right)^4 e^x = 5e^x\left(e^x+3\right)^4 \quad \square$$

e. *Product, Chain, and Power Rules.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x\tan\left(x^2\right)\right) = \left[\frac{d}{dx}x\right]\tan\left(x^2\right) + x\left[\frac{d}{dx}\tan\left(x^2\right)\right] \\ &= [1]\tan\left(x^2\right) + x\left[\sec^2\left(x^2\right)\cdot\frac{d}{dx}x^2\right] = \tan\left(x^2\right) + x\left[\sec^2\left(x^2\right)\cdot 2x\right] \\ &= \tan\left(x^2\right) + 2x^2\sec^2\left(x^2\right) \quad \square\end{aligned}$$

f. *Product and Power Rules.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2e^x\right) = \left[\frac{d}{dx}x^2\right]e^x + x^2\left[\frac{d}{dx}e^x\right] = [2x]e^x + x^2\left[e^x\right] = x(x+2)e^x \quad \blacksquare$$

2. Evaluate any four (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a.}\ \int \frac{x+1}{x^2+1}\,dx \quad \textbf{b.}\ \int_1^e \ln(x)\,dx \quad \textbf{c.}\ \int 6x^2\cos\left(x^3+\pi\right)\,dx$$

$$\textbf{d.}\ \int_0^1 x^2e^x\,dx \quad \textbf{e.}\ \int \frac{x+3}{x^2-9}\,dx \quad \textbf{f.}\ \int_0^\pi \sin(2x)\,dx$$

SOLUTIONS. **a.** *Subdivide the integrand and substitution.* We will be using the substitution $w = x^2+1$, so $dw = 2x\,dx$, and thus $x\,dx = \frac{1}{2}\,dw$, on part of the original integrand.

$$\begin{aligned}\int \frac{x+1}{x^2+1}\,dx &= \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \int \frac{1}{w}\cdot\frac{1}{2}\,dw + \arctan(x) \\ &= \frac{1}{2}\ln(w) + \arctan(x) + C = \frac{1}{2}\ln\left(x^2+1\right) + \arctan(x) + C \quad \square\end{aligned}$$

b. *Integration by parts using the dummy product trick.*

$$\int_1^e \ln(x)\,dx = \int_1^e 1\cdot\ln(x)\,dx \qquad \begin{array}{l}\text{Let } u = \ln(x) \text{ and } v' = 1,\\ \text{so } u' = \frac{1}{x} \text{ and } v = x.\end{array}$$

$$\begin{aligned}&= \ln(x)\cdot x\Big|_1^e - \int_1^e \frac{1}{x}\cdot x\,dx = x\ln(x)\Big|_1^e - \int_1^e 1\,dx \\ &= e\ln(e) - 1\ln(1) - x\Big|_1^e = e\cdot 1 - 1\cdot 0 - (e-1) = e - 0 - e + 1 = 1 \quad \square\end{aligned}$$

c. *Substitution.* We will use the substitution $w = x^3 + \pi$, so $dw = 3x^2\,dx$.

$$\int 6x^2 \cos\left(x^3+\pi\right)\,dx = \int 2\cdot 3x^2 \cos\left(x^3+\pi\right)\,dx = \int 2\cos(w)\,dw$$
$$= 2\sin(w) + C = 2\sin\left(x^3+\pi\right) + C \quad \square$$

d. *Integration by parts, twice.*

$$\int_0^1 x^2 e^x\,dx = x^2e^x\Big|_0^1 - \int_0^1 2xe^x\,dx \qquad \text{Where } u = x^2 \text{ and } v' = e^x, \text{ so } u' = 2x \text{ and } v = e^x.$$
$$= 1^2e^1 - 0^2e^0 - \left[2xe^x\Big|_0^1 - \int_0^1 2e^x\,dx\right] \qquad \text{Where } s = 2x \text{ and } t' = e^x, \text{ so } s' = 2 \text{ and } t = e^x.$$
$$= e - 0 - \left[2\cdot 1e^1 - 2\cdot 0e^0 - 2e^x\Big|_0^1\right] = e - \left[2e - 0 - \left(2e^1 - 2e^0\right)\right]$$
$$= e - [2e - 2e + 2] = e - [0+2] = e - 2 \quad \square$$

e. *Algebra and substitution.* We will use the substitution $w = x - 3$, so $dw = dx$, after simplifying the integrand.

$$\int \frac{x+3}{x^2-9}\,dx = \int \frac{x+3}{(x+3)(x-3)}\,dx = \int \frac{1}{x-3}\,dx$$
$$= \int \frac{1}{w}\,dw = \ln(w) + C = \ln(x-3) + C \quad \square$$

f. *The easy substitution.* We will use the substitution $w = 2x$, so $dw = 2\,dx$, and thus $dx = \frac{1}{2}\,dw$. We will change the limits as we go along: $\begin{matrix} x & w \\ 0 & 0 \\ \pi & 2\pi \end{matrix}$

$$\int_0^\pi \sin(2x)\,dx = \int_0^{2\pi} \sin(w)\,\frac{1}{2}\,dw = -\frac{1}{2}\cos(w)\Big|_0^{2\pi} = \left(-\frac{1}{2}\cos(2\pi)\right) - \left(-\frac{1}{2}\cos(0)\right)$$
$$= \left(-\frac{1}{2}\cdot 1\right) - \left(-\frac{1}{2}\cdot 1\right) = -\frac{1}{2} + \frac{1}{2} = 0 \quad \square$$

f. *Trigonometry and a harder substitution, but an easier integral.* We will use the substitution $z = \sin(x)$, so $dz = \cos(x)\,dx$, and change the limits as we go along: $\begin{matrix} x & z \\ 0 & 0 \\ \pi & 0 \end{matrix}$

$$\int_0^\pi \sin(2x)\,dx = \int_0^\pi 2\sin(x)\cos(x)\,dx = \int_0^0 2z\,dz = 0,$$

because we're integrating over a single point. ■

3. Do any four (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Compute $\lim_{x \to \infty} \frac{\ln(x)}{x}$.

b. Use the ε–δ definition of limits to verify that $\lim_{x \to -1} (2x+3) = 1$.

c. At what point (x, y) does the graph of $y = x^2$ have a tangent line with slope 4?

d. Sketch the region between $y = \cos(x)$ and $y = -\cos(x)$, for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, and find its area.

e. Let $f(x) = |x|$. Determine whether $f'(x)$ is defined at $x = 0$.

f. Suppose $f'(x) = \cos(x)$ and $f(0) = 2$. What is the function $f(x)$?

Solutions. **a.** *l'Hôpital's Rule.*

$$\lim_{x \to \infty} \frac{\ln(x) \to \infty}{x \quad \to \infty} = \lim_{x \to \infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1 \to 1}{x \to \infty} = 0 \quad \square$$

b. We need to show that for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x - (-1)| < \delta$, then it is guaranteed that $|(2x+3) - 1| < \varepsilon$. As usual, we try to reverse engineer the δ from the ε.

Suppose that we are given an $\varepsilon > 0$. Then

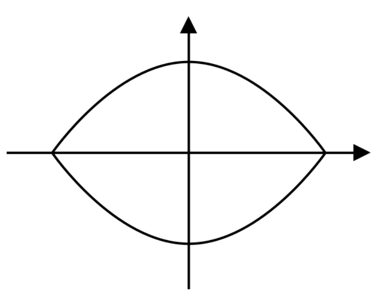
$$|(2x+3) - 1| < \varepsilon \iff |2x2| < \varepsilon \iff |2(x+1)| < \varepsilon$$
$$\iff 2\,|x - (-1)| < \varepsilon \iff |x - (-1)| < \frac{\varepsilon}{2}.$$

If we now set $\delta = \frac{\varepsilon}{2}$, then whenever $|x - (-1)| < \delta = \frac{\varepsilon}{2}$, we are guaranteed that $|(2x+3) - 1| < \varepsilon$ because we can run the process above backwards since each step is reversible.

Thus $\lim_{x \to -1} (2x+3) = 1$ by the ε–δ definition of limits. $\square$

c. *Where is the derivative equal to 4?* The slope of the tangent line at any given point on the graph is given by the derivative of the function at that point. $\frac{dy}{dx} = \frac{d}{dx}x^2 = 2x = 4$ exactly when $x = \frac{4}{2} = 2$. When $x = 2$, we have $y = x^2 = 2^2 = 4$. Thus the graph of $y = x^2$ has a tangent line with slope 4 at the point $(x, y) = (2, 4)$. $\square$

d. *Integrate to find the area.* Here is a crude sketch of the region:



It remains to compute the area of the region. Note that for x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, $-\cos(x) \leq 0 \leq \cos(x)$.

$$\begin{aligned}
\text{Area} &= \int_{-\pi/2}^{\pi/2} (\text{upper} - \text{lower})\, dx = \int_{-\pi/2}^{\pi/2} (\cos(x) - (-\cos(x)))\, dx \\
&= \int_{-\pi/2}^{\pi/2} 2\cos(x)\, dx = 2\sin(x)\Big|_{-\pi/2}^{\pi/2} = 2\sin\left(\frac{\pi}{2}\right) - 2\sin\left(-\frac{\pi}{2}\right) \\
&= 2 \cdot 1 - 2(-1) = 2 + 2 = 4 \quad \square
\end{aligned}$$

e. *Using the limit definition of the derivative.* The derivative of $f(x) = |x|$ is defined at $x = 0$ if the limit $f'(0) = \lim_{h \to 0} \frac{|0+h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}$ exists. However, since $\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$ and $\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{h}{h} = \lim_{h \to 0^-} 1 = 1$, and $-1 \neq 1$, the limit $\lim_{h \to 0} \frac{|h|}{h}$ does not exist. It follows that the derivative of $f(x) = |x|$ is not defined at $x = 0$. $\square$

f. Since $f'(x) = \cos(x)$, it follows that $f(x) = \int \cos(x)\, dx = \sin(x) + C$ for some constant C. Since $2 = f(0) = \sin(0) + C = 0 + C = C$, we must have that $f(x) = \sin(x) + 2$. ■

4. Find the domain, intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, maximum and minimum points, intervals of concavity, and inflection points of $f(x) = xe^x$. *[15]*

Solution. We run through the indicated checklist:

i. Domain. $f(x) = xe^x$ is defined (and continuous and differentiable) for all x, so its domain is "all x", *i.e.* $\mathbb{R} = (-\infty, \infty)$.

ii. Intercepts. $f(0) = 0e^0 = 0$, so $y = f(x)$ has y-intercept $y = 0$.

Since $e^x > 0$ for all x, $f(x) = xe^x = 0$ exactly when $x = 0$, so $y = f(x)$ has $x = 0$ as its only x-intercept.

iii. Vertical asymptotes. Since $f(x)$ is defined and continuous for all x, it cannot have any vertical asymptotes.

iv. Horizontal asymptotes. We check what $f(x)$ does as $x \to -\infty$ and as $x \to \infty$, with a little help from l'Hôpital's Rule. Note that $e^{-x} \to \infty$ as $x \to -\infty$.

$$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x \ \to \infty}{e^{-x} \to \infty} = \lim_{x \to -\infty} \frac{\frac{d}{dx} x}{\frac{d}{dx} e^{-x}} = \lim_{x \to -\infty} \frac{1 \ \to 1}{-e^{-x} \to -\infty} = 0^-$$

$$\lim_{x \to \infty} xe^x = \left(\lim_{x \to \infty} x\right)\left(\lim_{x \to \infty} e^x\right) = \infty \cdot \infty = \infty$$

Thus $f(x) = xe^x$ has $y = 0$ as a horizontal asymptote in the negative direction, which it approaches from below, but does not have a horizontal asymptote in the positive direction.

v. Increase, decrease, maxima, and minima. We first work out $f'(x)$.

$$f'(x) = \frac{d}{dx}(xe^x) = \left[\frac{d}{dx}x\right]e^x + x\left[\frac{d}{dx}e^x\right] = [1]e^x + x\,[e^x] = (x+1)e^x$$

Since $e^x > 0$ for all x, $f'(x) = (x+1)e^x = 0$ exactly when $x + 1 = 0$, *i.e.* exactly when $x = -1$. Moreover, $f'(x) < 0$ exactly when $x + 1 < 0$, *i.e.* exactly when $x < -1$, and $f'(x) > 0$ exactly when $x + 1 > 0$, *i.e.* exactly when $x > -1$. It follows that $f(x)$ is decreasing when $x < -1$ and increasing when $x > -1$, so it has a local (and absolute) minimum at $x = -1$. We summarize this in a table:

x	$(-\infty, -1)$	-1	$(-1, \infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

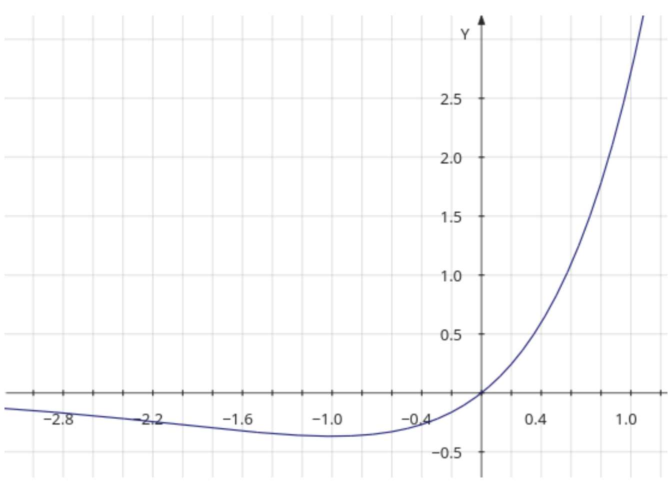
vi. Concavity and inflection points. We first work out $f''(x)$.

$$\begin{aligned} f''(x) &= \frac{d}{dx}f'(x) = \frac{d}{dx}((x+1)e^x) = \left[\frac{d}{dx}(x+1)\right]e^x + (x+1)\left[\frac{d}{dx}e^x\right] \\ &= [1]e^x + (x+1)\,[e^x] = (x+2)e^x \end{aligned}$$

Since $e^x > 0$ for all x, $f''(x) = (x+2)e^x = 0$ exactly when $x + 2 = 0$, *i.e.* exactly when $x = -2$. Moreover, $f''(x) < 0$ exactly when $x + 2 < 0$, *i.e.* exactly when $x < -2$, and $f''(x) > 0$ exactly when $x + 2 > 0$, *i.e.* exactly when $x > -2$. It follows that $f(x)$ is concave down when $x < -2$ and concave up when $x > -2$, so it has an inflection point at $x = -2$. We summarize this in a table:

x	$(-\infty, -2)$	-2	$(-2, \infty)$
$f''(x)$	$-$	0	$+$
$f(x)$	$\frown$	infl	$\smile$

vii. The graph. This wasn't actually asked for, but here it is anyway. Cheating just a bit, we drew the graph below using a program called KmPlot.



■

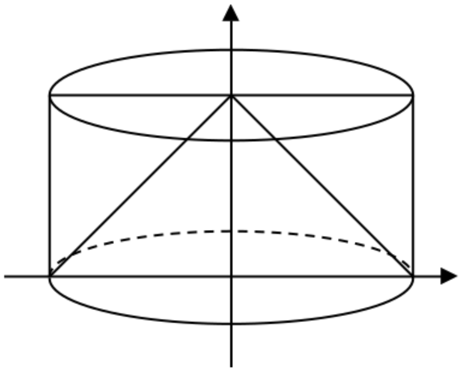
Part II. Do one (1) of **5–7**.

5. The region between $y = 4$ and $y = 4 - x$, where $0 \leq x \leq 4$, is revolved about the y-axis.

a. Sketch the resulting solid of revolution. *[2]*

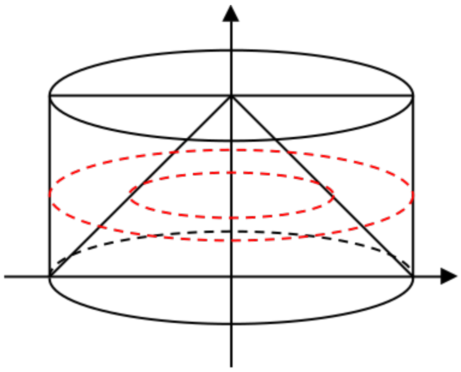
b. Find the volume of the solid. *[8]*

SOLUTIONS. **a.** Here is a crude sketch of the solid:



$\square$

b. *Disk/washer method.* Here is the crude sketch of the solid with a washer cross-section drawn in:



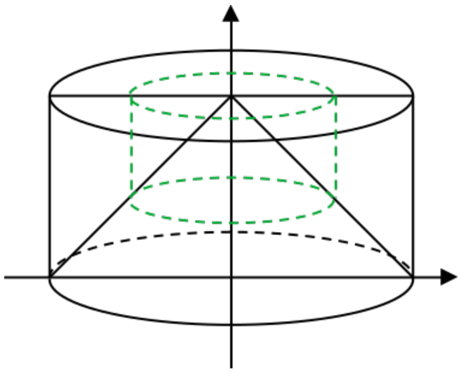
Since we revolved about the y-axis, the washers are horizontal and we ought to use y as the basic variable. Note that $0 \leq y \leq 4$ for the given region, and that $y = 4 - x \iff x = 4 - y$. The washer at y has outer radius $R = 4$ and inner radius $r = x = 4 - y$, so it has area

$$A(y) = \pi\left(R^2 - r^2\right) = \pi\left(4^2 - (4-y)^2\right) = \pi\left(16 - \left(16 - 8y + y^2\right)\right) = \pi\left(8y - y^2\right).$$

It follows that the volume of the solid is

$$\begin{aligned} V &= \int_0^4 A(y)\,dy = \int_0^4 \pi\left(8y - y^2\right)\,dy = \pi\left(\frac{8y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^4 = \pi\left(4y^2 - \frac{y^3}{3}\right)\Bigg|_0^4 \\ &= \pi\left(4 \cdot 4^2 - \frac{4^3}{3}\right) - \pi\left(4 \cdot 0^2 - \frac{0^3}{3}\right) = \pi\left(64 - \frac{64}{3}\right) - \pi \cdot 0 = \pi \cdot 64 \cdot \frac{2}{3} = \frac{128\pi}{3}. \quad \square \end{aligned}$$

b. *Cylindrical shell method.* Here is the crude sketch of the solid with a cylindrical shell cross-section drawn in:



Since we revolved about the y-axis, the shells are vertical and we ought to use x as the basic variable. The shell at x has radius $r = x - 0 = x$ and height $h = 4 - y = 4 - (4 - x) = x$, so it has area

$$A(x) = 2\pi rh = 2\pi x \cdot x = 2\pi x^2.$$

It follows that the volume of the solid is

$$V = \int_0^4 A(x)\,dx = \int_0^4 2\pi x^2\,dx = 2\pi \frac{x^3}{3}\Big|_0^4 = 2\pi\frac{4^3}{3} - 2\pi\frac{0^3}{3} = 2\pi\frac{64}{3} - 2\pi \cdot 0 = \frac{128\pi}{3}. \quad \square$$

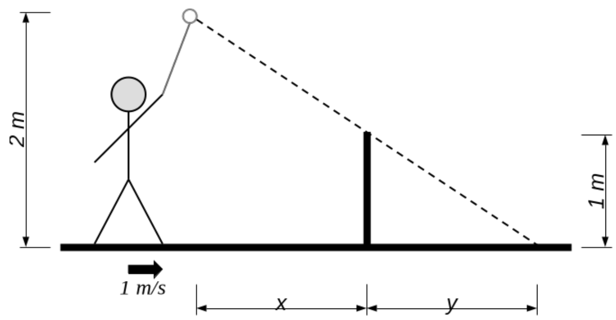
b. *Geometry.* The solid is a cylinder of height and radius 4 with a cone of base radius 4 and height 4 cut out of it. The volume of the cylinder is $\pi r^2 h = \pi \cdot 4^2 \cdot 4 = 64\pi$ and the volume of the cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3} \cdot \pi \cdot 4^2 \cdot 4 = \frac{64\pi}{3}$, so the volume of the solid is $64\pi - \frac{64\pi}{3} = 64\pi \cdot \frac{2}{3} = \frac{128\pi}{3}$. ■

6. It is night. Meredith Stick, who is 1.5 m tall, walks slowly at 1 m/s on level ground, holding a lamp on a stick 2 m above the ground. Meredith is moving straight towards a 1 m tall fence post, which casts a shadow on the ground in the light from the lamp.

a. Draw a diagram of this setup. *[2]*

b. How is the length of this shadow changing at the instant that Meredith is 4 m from the post? *[8]*

SOLUTIONS. **a.** Here is a diagram of the situation:



□

b. As in the diagram, let x be the horizontal distance between the lamp and the fence post, and let y be the length of the shadow. We are given that $\frac{dx}{dt} = -1\ m/s$, since Meredith stick is walking *towards* the post at $1\ m/s$, and we wat to work out $\frac{dy}{dt}$,

The shadow is the base, with length y, and the post the height, with length 1, of a right triangle. Consider also the right triangle whose base is the horizontal distance from the lamp to the tip of the shadow, with length $x + y$, and whose height is the elevation of the lamp above the ground, with length 2. These two triangles are similar because they share a common angle at the tip of the shadow and each has a right angle as one of its other angles.

Since the two triangles have the same proportions, $\frac{y}{1} = \frac{x+y}{2}$. This means that

$$\frac{y}{1} = \frac{x+y}{2} = \frac{x}{2} + \frac{y}{2} \implies \frac{x}{2} = y - \frac{y}{2} = \frac{y}{2} \implies x = y.$$

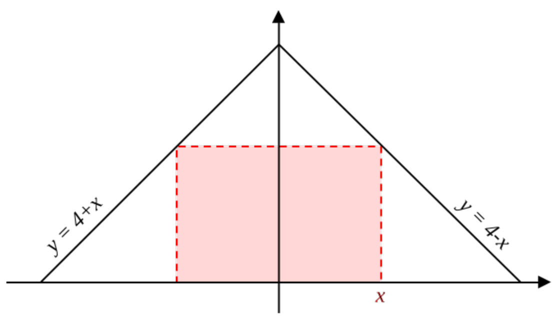
It follows that $\frac{dy}{dt} = \frac{dx}{dt} = -1\ m/s$, *i.e.* the length of the shadow cast by the post in the light from the lamp is shrinking at a rate of $1\ m/s$, at every instant, including when $x = 4\ m$. ■

7. A rectangle has its base on the part of the x-axis with $-4 \le x \le 4$, and its upper corners on the lines $y = 4 + x$ and $y = 4 - x$, respectively.

a. Draw a diagram of this setup. *[2]*

b. What is the maximum possible area of such a rectangle? *[8]*

SOLUTIONS. **a.** Here is a crude diagram of this setup:



□

b. For each x with $0 \le x \le 4$ we get a distinct rectangle with base $b = x - (-x) = 2x$ and height $h = 4 - x$. (Note that there is no need to look at negative xs because x and $-x$ give us the same rectangle.) Thus the rectangle at x has area $A(x) = bh = 2x(4-x) = 8x - 2x^2$. We wish to maximize the area for $0 \le x \le 4$.

$$A'(x) = \frac{d}{dx}\left(8x - 2x^2\right) = 8 - 4x$$

Thus $A'(x) = 8 - 4x = 0$ exactly when $x = \frac{8}{4} = 2$, which critical point is between 0 and 4. Note also that $A'(x) = 8 - 4x < 0$ exactly when $x > 2$, and that $A'(x) = 8 - 4x > 0$

exactly when $x < 2$. It follows that $A(x)$ is increasing to the left of the critical point $x = 2$ and decreasing to the right of the same critical point, and thus $x = 2$ is a maximum.

Since $A(x)$ is defined and differentiable for $0 \leq x \leq 4$, it follows that the maximum area of a rectangle as in the setup is $A(2) = 8 \cdot 2 - 2 \cdot 2^2 = 16 - 8 = 8$. As a sanity check, we see what the area is at the endpoints. $A(0) = 8 \cdot 0 - 2 \cdot 0^2 = 0$ and $A(4) = 8 \cdot 4 - 2 \cdot 4^2 = 32 - 32 = 0$, both of which are smaller than $A(2) = 8$, confirming that this is the maximum area. ■

[Total = 85]

Part III. *Here be bonus points!* Do none, or one, or both of the following questions.

$\sqrt{\mathbf{64}}$**.** Suppose you know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (Which is true.) What does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$ then have to be? *[1]*

Solution. Nope, not saying! :-) ■

$\sqrt{\mathbf{81}}$**.** Write a haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

Solution. The haiku above does touch on arithmetic ... ■

Rest, relax, and enjoy the rest of the summer!