Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals

TRENT UNIVERSITY, Summer 2025 (S62)

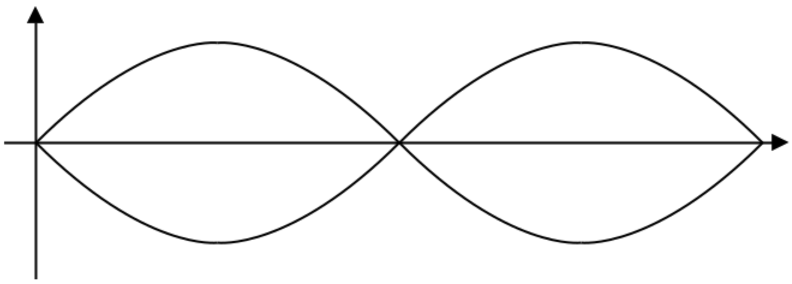
Solutions to Assignment #5 – A Solid of Revolution

Due on Friday, 25 July.

Consider the region between $y = \sin(x)$ and $y = -\sin(x)$, where $0 \leq x \leq 2\pi$.

1. Sketch or plot this region. *[1]*

SOLUTION. Here is a crude sketch of the region:



$\square$

2. Find the area of this region. *[2]*

SOLUTION. Note that $-\sin(x) \leq 0 \leq \sin(x)$ when $0 \leq x \leq \pi$ and $\sin(x) \leq 0 \leq -\sin(x)$ when $\pi \leq x \leq 2\pi$. It follows that the area between the two curves for $0 \leq x \leq 2\pi$ is

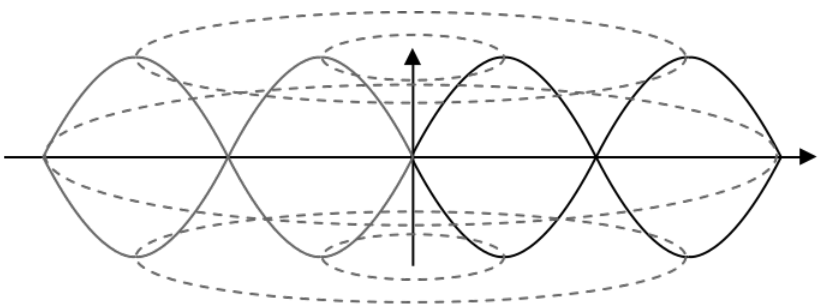
$$\begin{aligned}
\text{Area} &= \int_0^{\pi} (\text{upper} - \text{lower})\, dx \\
&= \int_0^{\pi} (\sin(x) - (-\sin(x)))\, dx + \int_{\pi}^{2\pi} (-\sin(x) - \sin(x))\, dx \\
&= \int_0^{\pi} 2\sin(x)\, dx - \int_{\pi}^{2\pi} 2\sin(x)\, dx = \left(-2\cos(x)\right)\Big|_0^{\pi} - \left(-2\cos(x)\right)\Big|_{\pi}^{2\pi} \\
&= (-2\cos(\pi)) - (-2\cos(0)) - [(-2\cos(2\pi)) - (-2\cos(\pi))] \\
&= -(-2) - (-2) - [(-2) - (-(-2))] = 2 + 2 - [-2 - 2] = 4 - [-4] = 8 \quad \square
\end{aligned}$$

NOTE. Of course, we could have handed off working out the integration to SageMath once we set it up.

Consider the solid obtained by revolving the given region about the y-axis.

3. Sketch or plot this solid. *[1]*

SOLUTION. Here is a crude sketch of the solid:

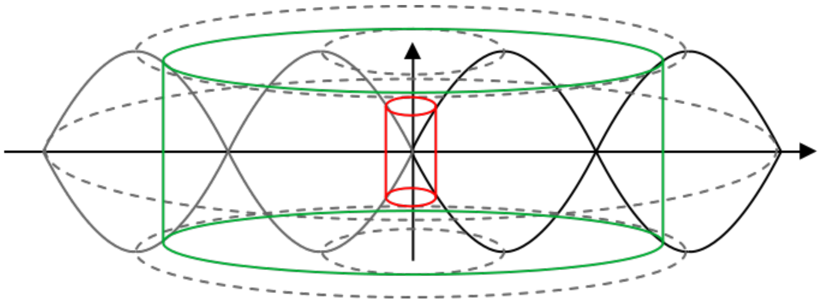


$\square$

4. Compute the volume of this solid using the cylindrical shell method. *[4]*

HINT. See the end Section 9.3 of the textbook for an example of this method. There are also a number of examples among the work with solutions on the archive page.

SOLUTION. Here is the same crude sketch of the solid with a couple of cylindrical shells drawn in:



Note that for x with $0 \leq x \leq \pi$, the cylindrical shell at x has radius $r = x - 0 = x$ and height $h = \sin(x) - (-\sin(x)) = 2\sin(x)$, and hence has area $A(x) = 2\pi rh = 4\pi x \sin(x)$. On the other hand, for x with $\pi \leq x \leq 2\pi$, the cylindrical shell at x has radius $r = x - 0 = x$ and height $h = -\sin(x) - \sin(x) = -2\sin(x)$, and hence has area $A(x) = 2\pi rh = -4\pi x \sin(x)$. (Keep in mind that $\sin(x)$ is negative between π and 2π.) It follows that the volume of this solid of revolution is given by:

$$
\begin{aligned}
\text{Volume} &= \int_0^{2\pi} A(x)\,dx = \int_0^{\pi} 4\pi x \sin(x)\,dx + \int_{\pi}^{2\pi} \left(-4\pi x \sin(x)\right)\,dx \\
&= \int_0^{\pi} 4\pi x \sin(x)\,dx - \int_{\pi}^{2\pi} 4\pi x \sin(x)\,dx \qquad \text{Integration by parts with } u = x \text{ and } v' = \sin(x) \text{, so } u' = 1 \text{ and } v = -\cos(x). \\
&= 4\pi \left[\left(-x\cos(x)\right)\Big|_0^{\pi} - \int_0^{\pi} \left(-\cos(x)\right)\,dx \right] \\
&\qquad - 4\pi \left[\left(-x\cos(x)\right)\Big|_{\pi}^{2\pi} - \int_{\pi}^{2\pi} \left(-\cos(x)\right)\,dx \right] \\
&= 4\pi \left[\left(-x\cos(x)\right)\Big|_0^{\pi} + \int_0^{\pi} \cos(x)\,dx \right] - 4\pi \left[\left(-x\cos(x)\right)\Big|_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos(x)\,dx \right] \\
&= 4\pi \left[(-\pi\cos(\pi)) - (-0\cos(0)) + \sin(x)\Big|_0^{\pi} \right] \\
&\qquad - 4\pi \left[(-2\pi\cos(2\pi)) - (-\pi\cos(\pi)) + \sin(x)\Big|_{\pi}^{2\pi} \right] \\
&= 4\pi \left[-\pi \cdot (-1) + 0 \cdot 1 + \sin(\pi) - \sin(0) \right] \\
&\qquad - 4\pi \left[-2\pi \cdot 1 + \pi \cdot (-1) + \sin(2\pi) - \sin(\pi) \right] \\
&= 4\pi[\pi + 0 + 0 - 0] - 4\pi[-2\pi - \pi + 0 - 0] = 4\pi^2 - 4\pi[-3\pi] \\
&= 4\pi^2 + 12\pi^2 = 16\pi^2 \quad \square
\end{aligned}
$$

5. Compute the volume of this solid by hand without using calculus. *[2]*

HINT. You are allowed to look things up ...

SOLUTION. There is a theorem from geometry that makes this pretty easy:

> PAPPUS' CENTROID THEOREM. *If a plane region is revolved around an external axis, the volume of the resulting solid of revolution is equal to the product of the area of the region and the distance travelled by the centroid of the region.**

The centroid of a region is its centre of gravity (assuming constant density). In the present case, the symmetry of the given region means its centroid is the point where the two parts of the region touch, *i.e.* the point $(\pi, 0)$. This point travels through a circle with radius π when the region is revolved about the y-axis, so it covers a distance of $2\pi r = 2\pi \cdot \pi = 2\pi^2$. By question **2** the region has area 8, so by Pappus' Centroid Theorem, it follows that the volume of the solid of revolution is $8 \cdot 2\pi^2 = 16\pi^2$. ■

* This theorem appears in the work of Pappus of Alexandria (*c.* 290 – 350 A.D.), and was rediscovered by Johannes Kepler (1571 – 1630 A.D.) and Paul Guldin (1577 – 1643 A.D.). The relevant part of Pappus' work apparently did not appear in print, and hence get wide circulation, until 1659. By the way, Pappus stated another theorem, also often called Pappus' Centroid Theorem, that lets you compute the surface area of a solid of revolution in a similar way.