Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2025 (S62)

Solutions to Assignment #4 – Integration
Due on Friday, 18 July.

Please at least skim Sections 1.12, 4.8, and 4.19 of *Sage for Undergraduates*, which cover the basics of the `integral`, `limit`, and `sum` commands, respectively, before tackling this assignment. Please note that SageMath uses `oo` when you need to indicate ∞ in these commands. For the Right-Hand Rule, please take a look at the handout *Right-Hand Rule Riemann Sums*, which you can find in the *Textbook and Handouts* folder in the *Course Content* section on Blackboard, or on the archive page. You can find the basics of what definite and indefinite integrals (*i.e.* antiderivatives) are in Chapter 7 of the textbook.

In questions **1**–**5**, let $f(x) = x^2 - 1$.

1. Use SageMath to solve the first-order linear differential equation $\frac{dy}{dx} = f(x)$ to find the (general) antiderivative of $f(x)$. *[1]*

SOLUTION. Here we go:

```
[1]: f = function('f')(x)
     y = function('y')(x)
     f(x) = x^2 - 1

[2]: desolve( diff(y,x) == f(x), y )

[2]: 1/3*x^3 + _C - x
```

That is, the general antiderivative of $f(x) = x^2 - 1$ is $y = \frac{x^3}{3} - x + C$. ■

2. Use SageMath to find the (general) antiderivative of $f(x)$ with the `integral` command. *[2]*

SOLUTION. Here we go:

```
[3]: y = integral( f(x), x )
     show(y)

     1/3*x^3 - x
```

SageMath gives us $\frac{x^3}{3} - x$, without the generic constant we got in question **1**, so it isn't the most general possible antiderivative. ■

3. Use the antiderivative you found in answering question **1** or **2** to compute the definite integral $\int_{-1}^{2} f(x)\, dx$. *[1]*

SOLUTION. Being lazy, we let SageMath do the work, using the antiderivative it gave us in question **2**.

```
[4]: y(2) - y(-1)
```

```
[4]: 0
```

■

NOTE. Of course, the smart thing would have been to ask SageMath to compute the definite integral $\int_{-1}^{2} f(x)\,dx$ directly:

```
[5]: integral( f(x), x, -1, 2 )
```

```
[5]: 0
```

4. Compute the definite integral $\int_{-1}^{2} f(x)\,dx$ using the Right-Hand Rule, using SageMath to do the actual calculation. *[3]*

SOLUTION. Here's a code snippet that does the job and could be used for any similar calculation by modifying a, b, and $f(x)$.

```
[6]: var('n')
     var('i')
     R = function('R')(n)
     a = -1
     b = 2
     R(n) = (b-a)/n * sum( f( a + i*(b-a)/n), i, 1, n )
     limit( R(n), n = oo )
```

```
[6]: 0
```

■

5. Compute the definite integral $\int_{-1}^{2} f(x)\,dx$ using the Right-Hand Rule by hand. *[3]*

HINT FOR **5**. You may make use of the summation formulas $\sum_{i=1}^{n} 1 = n$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

SOLUTION. We plug $f(x) = x^2 - 1$, $a = -1$, and $b = 2$ into the generic Right-Hand Rule formula

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left[\frac{b-a}{n} \cdot \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \right]$$

and do a lot of algebra ... Here goes:

$$
\begin{aligned}
\int_{-1}^{2} \left(x^2 - 1\right)\, dx &= \lim_{n\to\infty} \left[\frac{2-(-1)}{n} \cdot \sum_{i=1}^{n} \left(\left(-1 + i \cdot \frac{2-(-1)}{n} \right)^2 - 1 \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n} \cdot \sum_{i=1}^{n} \left(\left(-1 + i \cdot \frac{3}{n} \right)^2 - 1 \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n} \cdot \sum_{i=1}^{n} \left((-1)^2 - 2\frac{3i}{n} + \left(\frac{3i}{n} \right)^2 - 1 \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n} \cdot \sum_{i=1}^{n} \left(1 - \frac{6i}{n} + \frac{9i^2}{n^2} - 1 \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{3}{n} \cdot \sum_{i=1}^{n} \left(\frac{9}{n^2} i^2 - \frac{6}{n} i \right) \right] \\
&= \lim_{n\to\infty} \left[\left(\frac{3}{n} \cdot \sum_{i=1}^{n} \frac{9}{n^2} i^2 \right) - \left(\frac{3}{n} \cdot \sum_{i=1}^{n} \frac{6}{n} i \right) \right] \\
&= \lim_{n\to\infty} \left[\left(\frac{3}{n} \cdot \frac{9}{n^2} \cdot \sum_{i=1}^{n} i^2 \right) - \left(\frac{3}{n} \cdot \frac{6}{n} \cdot \sum_{i=1}^{n} i \right) \right] \\
&= \lim_{n\to\infty} \left[\frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{18}{n^2} \cdot \frac{n(n+1)}{2} \right] \\
&= \lim_{n\to\infty} \left[\frac{27}{6} \cdot \frac{2n^3 + 3n^2 + n}{n^3} - \frac{18}{2} \cdot \frac{n^2 + n}{n^2} \right] \\
&= \lim_{n\to\infty} \left[\frac{9}{2} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) - 9 \left(1 + \frac{1}{n} \right) \right] \\
&= \frac{9}{2} (2 + 0 + 0) - 9 (1 + 0) = 9 - 9 = 0
\end{aligned}
$$

Whew! ■