Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals

Trent University, Summer 2025 (S62)

Solutions to Assignment #3
A Very Few Differential Equations

Due on Friday, 11 July.

Since the midterm test will be written while this assignment is live, it is not intended to be all that challenging, though it does involve learning how to take derivatives and solve equations involving them. If you haven't already, please read Subsections 4.22.1 and 4.22.2 of *Sage for Undergraduates*, before tackling this assignment.

A *first-order linear differential equation* is one of the form $a(x) \cdot \frac{dy}{dx} + b(x) \cdot y = c(x)$, which is to be solved for the unknown function(s) y of x that satisfy the equation. It is a differential equation because it involves the derivative of the unknown function; it is first-order because it only involves the first derivative of the unknown function; it is linear because both the unknown function and its derivative occur by themselves in a sum with coefficients that are at most some given functions of x. The equation is further said to be *homogeneous* if the right-hand side function, $c(x)$, is just 0. A requirement that y have a specified value when x has some specified value is said to be an *initial condition* and generally picks out a particular solution from the multiple possible solutions to a given differential eqution.

Differential equations occur in a lot of applications and so have been, and continue to be, extensively studied. Sadly, a lot of them are hard to solve (and some are impossible to solve in nice terms), but linear differential equations are among the happy exceptions. In this assignment you will develop a formula for the solutions of a (narrow!) class of homogeneous first-order linear differential equations with a particular initial condition with the help of SageMath.

1. Use SageMath to solve the homogeneous first-order linear differential equation $\frac{dy}{dx} + x^n y = 0$ with initial condition $(x, y) = (0, 1)$ for:

a. $n = 0$ **b.** $n = 1$ **c.** $n = 2$ **d.** $n = 3$ **e.** $n = 4$ *[5 = 5×1 each]*

Solutions. **a.** Just for fun, we'll look at the general solution, without any initial condition, first. Here goes:

```
[1]: y = function('y')(x)
     desolve( diff(y,x) + y == 0, y )
```

```
[1]: _C*e^(-x)
```

That is, the general solution to the differential equation $\frac{dy}{dx} + y = 0$ is $y = Ce^{-x}$, where C could be any constant.

Once more, with the asked for initial condition:

```
[2]: desolve( diff(y,x) + y == 0, y, ics=[0,1] )
```

```
[2]: e^(-x)
```

That is, the solution to the differential equation $\frac{dy}{dx} + y = 0$ with initial condition $(x, y) = (0, 1)$ is $y = e^{-x}$. This is the general solution with $C = 1$. □

b. Here goes:

```
[3]: desolve( diff(y,x) + x*y == 0, y, ics=[0,1] )
```

```
[3]: e^(-1/2*x^2)
```

That is, the solution to the differential equation $\frac{dy}{dx} + xy = 0$ with initial condition $(x, y) = (0, 1)$ is $y = e^{-x^2/2}$. □

c. Here goes:

```
[4]: desolve( diff(y,x) + x^2*y == 0, y, ics=[0,1] )
```

```
[4]: e^(-1/3*x^3)
```

That is, the solution to the differential equation $\frac{dy}{dx} + x^2y = 0$ with initial condition $(x, y) = (0, 1)$ is $y = e^{-x^3/3}$. □

d. Here goes:

```
[5]: desolve( diff(y,x) + x^3*y == 0, y, ics=[0,1] )
```

```
[5]: e^(-1/4*x^4)
```

That is, the solution to the differential equation $\frac{dy}{dx} + x^3y = 0$ with initial condition $(x, y) = (0, 1)$ is $y = e^{-x^4/4}$. □

e. Here goes:

```
[6]: desolve( diff(y,x) + x^4*y == 0, y, ics=[0,1] )
```

```
[6]: e^(-1/5*x^5)
```

That is, the solution to the differential equation $\frac{dy}{dx} + x^4y = 0$ with initial condition $(x, y) = (0, 1)$ is $y = e^{-x^5/5}$. ■

2. Use the answers you obtained in solving question **1** to guess a solution to $\frac{dy}{dx} + x^n y = 0$ with initial condition $(x, y) = (0, 1)$ in terms of x and the integer $n \geq 0$. Verify, by hand, that your solution to thie equation works. *[3]*

SOLUTION. The pattern is pretty easy to spot here: for $n \geq 0$, $\frac{dy}{dx} + x^n y = 0$ with initial condition $(x, y) = (0, 1)$ will have the solution $y = e^{-x^{n+1}/(n+1)}$. We check that this solution does satisfy the differential equation with the given initial condition.

$$\begin{aligned}\frac{dy}{dx} + x^n y &= \frac{d}{dx} e^{-x^{n+1}/(n+1)} + x^n e^{-x^{n+1}/(n+1)} \\ &= e^{-x^{n+1}/(n+1)} \cdot \frac{d}{dx}\left(\frac{-x^{n+1}}{n+1}\right) + x^n e^{-x^{n+1}/(n+1)} \\ &= e^{-x^{n+1}/(n+1)} \cdot \frac{-(n+1)x^n}{n+1} + x^n e^{-x^{n+1}/(n+1)} \\ &= -x^n e^{-x^{n+1}/(n+1)} + x^n e^{-x^{n+1}/(n+1)} = 0\end{aligned}$$

Also, $y = e^{-0^{n+1}/(n+1)} = e^0 = 1$, so this solution satisfies both the differential equation and the given initial condition. ■

3. What happens if you set $n = -1$? Is there a solution for the initial condition $(x, y) = (0, 1)$? Are there any other initial conditions for which you get a solution? *[2]*

SOLUTION. There is a general solution,

```
[4]: y = function('y')(x)
     desolve( diff(y,x) + x^(-1)*y == 0, y )
```

```
[4]: _C/x
```

namely $y = \frac{C}{x}$, where C is any constant. However, this solution is undefined at 0.

If you plug the initial conditions $(x, y) = (0, 1)$ into the differential equation and turn SageMath loose you get a long string of error messages:

```
[5]: desolve( diff(y,x) + x^(-1)*y == 0, y, ics=[0,1] )
```

```
     ␣
↪---------------------------------------------------------------------------

     RuntimeError                              Traceback (most recent call␣
↪last)

                              ⋮

     TypeError: ECL says: Error executing code in Maxima: expt: undefined: 0␣
↪to a negative exponent.
```

The one at the end finally tells you that a denominator of 0 is undefined.

However, initial conditions with $x \neq 0$ may be satisfiable by solutions to the given differential equation. For example, if $(x, y) = (1, 1)$, SageMath gives us:

[6]:
```
desolve( diff(y,x) + x^(-1)*y == 0, y, ics=[1,1] )
```

[6]:
```
1/x
```

That is, $y = \dfrac{1}{x}$ is a solution to the differential equation $\dfrac{dy}{dx} + x^{-1}y = 0$ that satisfies the initial condition $(x, y) = (1, 1)$. ■