Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2025 (S62)

Solutions to Assignment #2 – Implicitly Defined Curves
Due on Friday, 4 July.

Since this assignment will go live near the beginning of the extra-long Canada Day weekend, it is not intended to be all that challenging or introduce anything really new. If you haven't already, please read Section 3.4 of *Sage for Undergraduates*, which is about plotting curves implicitly defined by equations in x and y, before tackling this assignment.

Here are five well-studied curves that are defined implicitly. Each equation includes a fixed *parameter* a and perhaps also b; changing the value of a parameter modifies the curve in some way.

Cardioid: $\left(x^2+y^2\right)^2+4ax\left(x^2+y^2\right)-4a^2y^2=0$ [For $a>0$ only.]

Nephroid: $\left(x^2+y^2-4a^2\right)^3=108a^4y^2$

Deltoid: $\left(x^2+y^2\right)^2+18a^2\left(x^2+y^2\right)-27a^4=8a\left(x^3-3xy^2\right)$

Lemniscate of Bernoulli: $\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)$ [Jakob Bernoulli, that is.]

Limaçon: $\left(x^2+y^2-ax\right)^2=b^2\left(x^2+y^2\right)$

1. Use SageMath to plot each of the following curves. You will probably need to experiment a bit with the ranges of x and y in each case to ensure that your plot displays the whole curve.

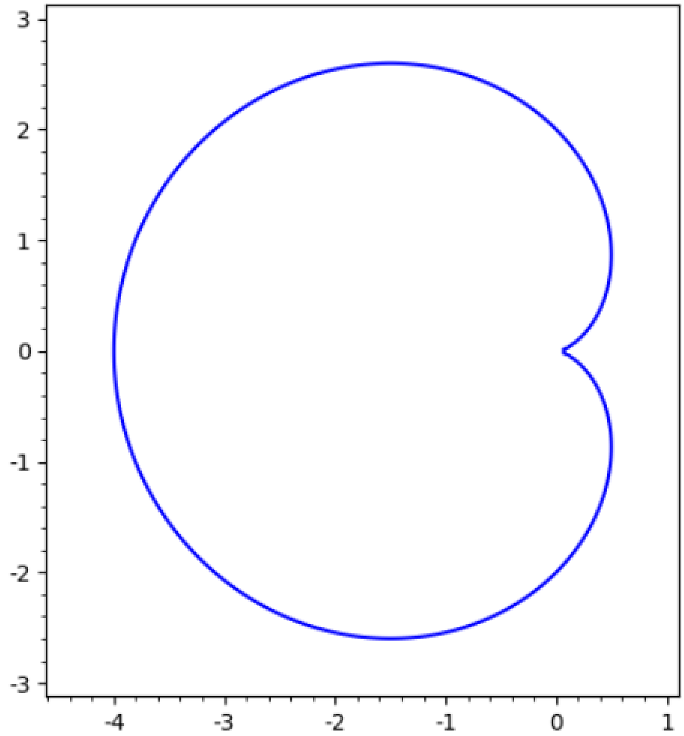
a. The cardioid with parameter $a=1$. *[1]*
b. The nephroid with parameter $a=1$. *[1]*
c. The deltoid with parameter $a=1$. *[1]*
d. The lemniscate of Bernoulli with parameter $a=1$. *[1]*
e. The limaçon with parameters $a=1$ and $b=\frac{1}{2}$. *[1]*
f. The limaçon with parameters $a=1$ and $b=1$. *[1]*
g. The limaçon with parameters $a=1$ and $b=2$. *[1]*

SOLUTIONS. Here we go, at one plot per page:

a.

[1]:
```
# Cardioid with parameter a=1.
var('y')
a=1
implicit_plot( (x^2 + y^2)^2 + 4*a*x*( x^2 + y^2 ) - 4*a^2*y^2 == 0, (x,-4.
↪5,1), (y,-3,3) )
```

[1]:

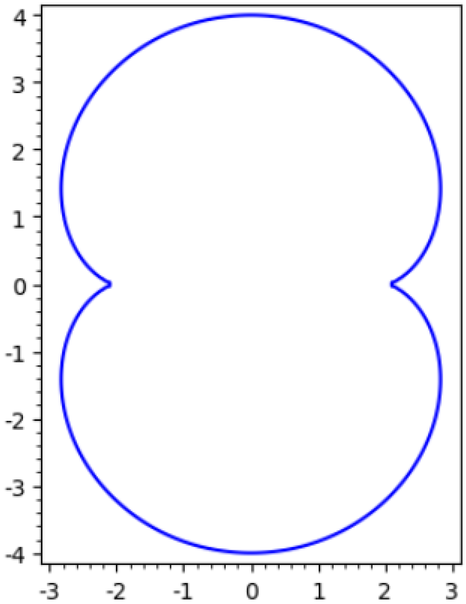


□

b.

[2]:
```
# Nephroid with parameter a=1.
a=1
implicit_plot( (x^2 + y^2 - 4*a^2)^3 == 108*a^4*y^2, (x,-3,3), (y,-4,4) )
```

[2]:

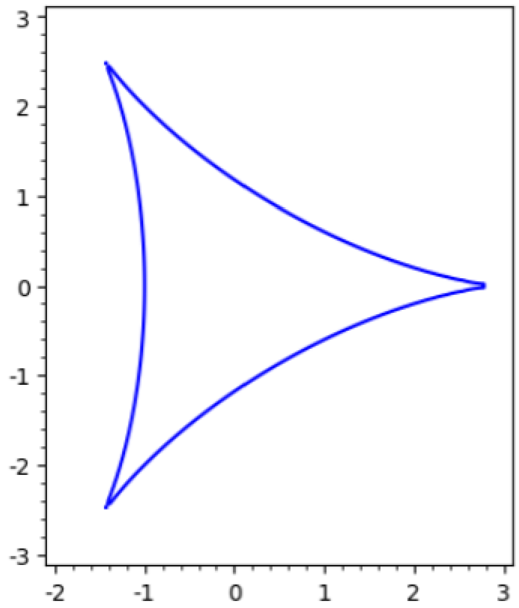


□

c.

[3]:
```
# Deltoid with parameter a=1.
a=1
implicit_plot( (x^2 + y^2)^2 + 18*a^2*( x^2 + y^2 ) - 27*a^4 == 8*a*( x^3 -
    3*x*y^2 ), (x,-2,3), (y,-3,3) )
```

[3]:

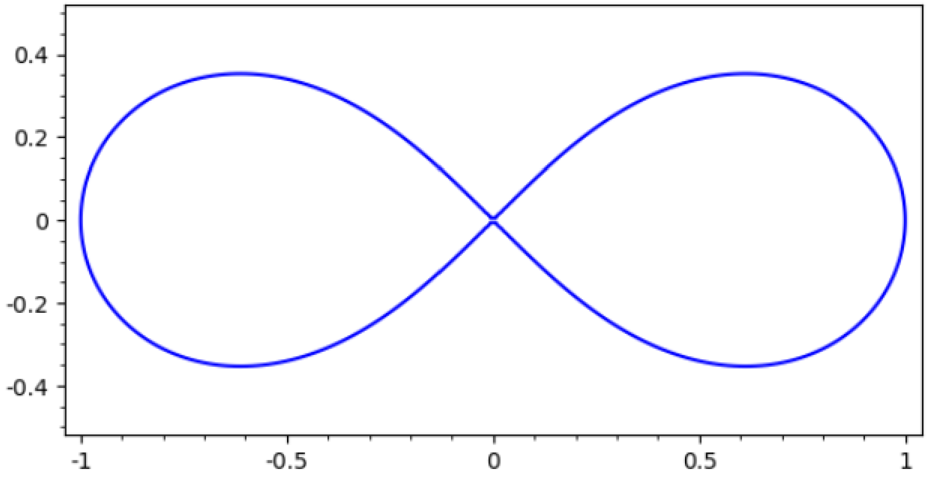


□

d.

[4]:
```
# Lemniscate of Bernoulli with parameter a=1.
a=1
implicit_plot( (x^2 + y^2)^2 == a^2*(x^2 - y^2), (x,-1,1), (y,-0.5,0.5) )
```

[4]:

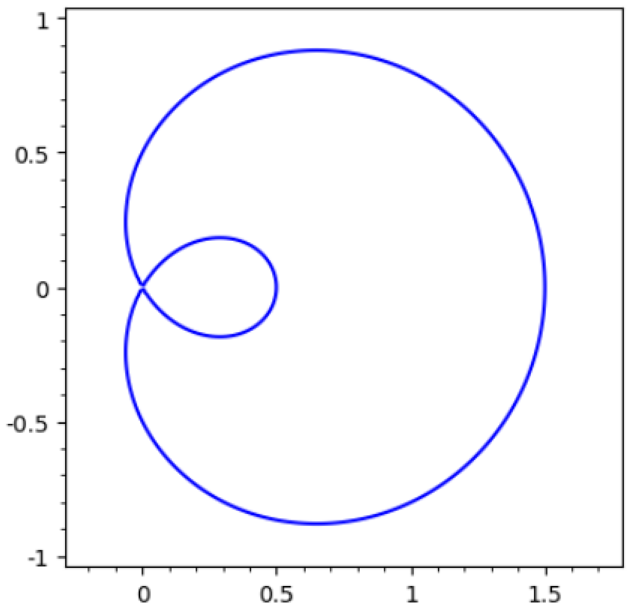


□

e.

[5]:
```
# Limaçon with parameters a=1 and b=1/2.
a=1
b=1/2
implicit_plot( (x^2 + y^2 - a*x)^2 == b^2*(x^2 + y^2), (x,-0.25,1.75), (y,-1,1)␣
↪)
```

[5]:



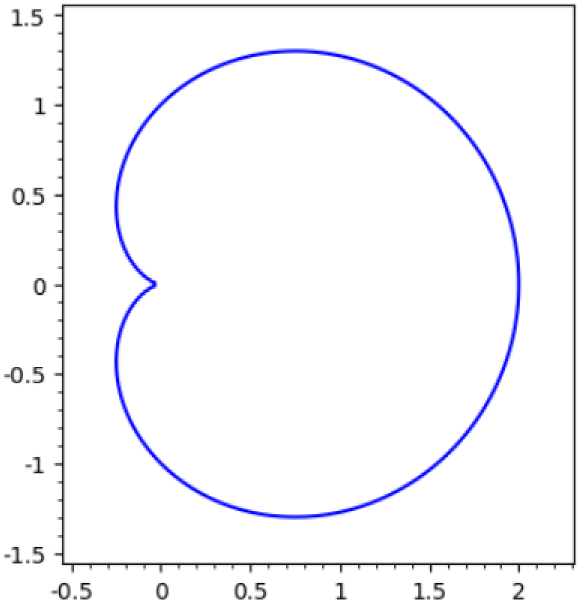
□

f.

[6]:

```
# Limaçon with parameters a=1 and b=1.
a=1
b=1
implicit_plot( (x^2 + y^2 - a*x)^2 == b^2*(x^2 + y^2), (x,-0.5,2.25), (y,-1.5,1.
↪5) )
```

[6]:

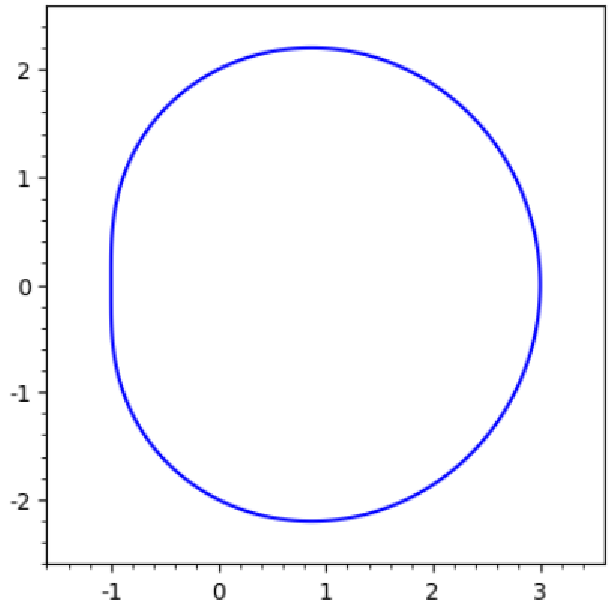


g.

[7]:

```
# Limaçon with parameters a=1 and b=2.
a=1
b=2
implicit_plot( (x^2 + y^2 - a*x)^2 == b^2*(x^2 + y^2), (x,-1.5,3.5), (y,-2.5,2.
↪5) )
```

[7]:



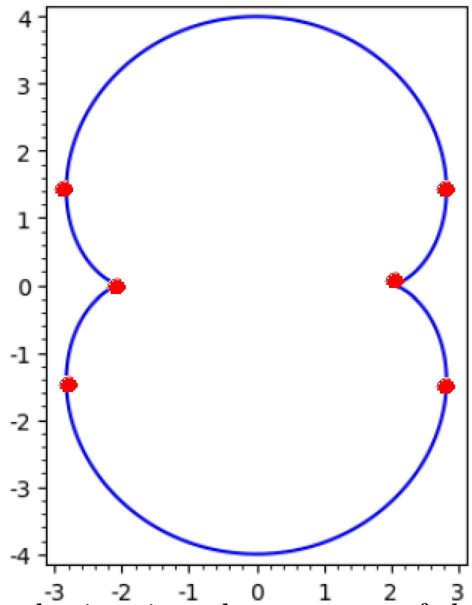
■

2. Consider the plot of the nephroid you (hopefully!) obtained in part **b** of question **1**.

a. If you decompose the graph of the nephroid into pieces that are each a graph of some continuous function $y = f(x)$ (with limited domain in each case), how many pieces do you need? Indicate what these pieces are with a suitable sketch or sketches. *[2]*

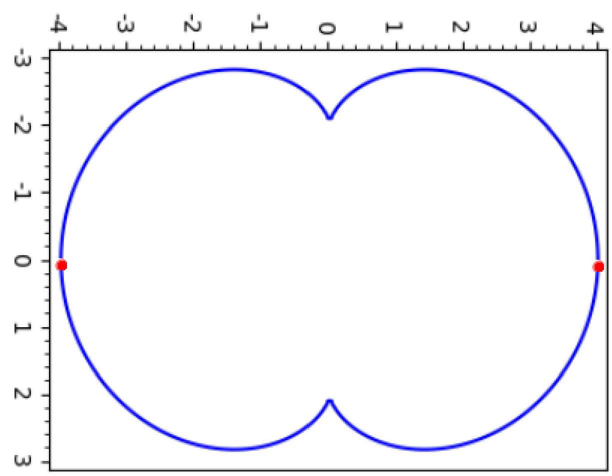
b. If you decompose the graph of the nephroid into pieces that are each a graph of some continuous function $x = g(y)$ (with limited domain in each case), how many pieces do you need? Indicate what these pieces are with a suitable sketch or sketches. *[1]*

SOLUTIONS. **a.** To be the graph of a function, a curve must pass the vertical line test, that each vertical line intersecting the curve does so only once. If you cut up the nephroid at the red dots in the picture below, each piece will pass the vertical line test, and so is the graph of some function.



There are six pieces. Since each piece is as large a part of the nephroid as can pass the vertical line test – look at how it curves! – one needs at least six pieces. □

b. Turning the plot on its side to make it easier to think of x as a function of y, we can use the same reasoning as in the solution to part **a** above to see that we need only two pieces.



■