Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals

TRENT UNIVERSITY, Summer 2025 (S62)

Solutions to Assignment #1 – Solving With SageMath

In all that follows, let
$$f(x) = \ln\left(x + \sqrt{x^2+1}\right).$$

Most of this assignment is concerned with finding $f^{-1}(x)$, the inverse function of $f(x)$, *i.e.* the function such that $y = f(x) \iff x = f^{-1}(y)$, or $f^{-1}(f(x)) = x$, at least when everything is duly defined.

Please recall that SageMath calls the natural logarithm function `log`. Also, take a peek at Section 3.4 of *Sage for Undergraduates*, by Gregory Bard, which has to do have to do with plotting graphs of implicitly defined curves, before doing question **1c**, and Section 1.8, which has to do with solving equations, before doing question **2**.

1. a. What is the domain of $f(x)$? *[0.5]*

b. Use SageMath to plot $y = f(x)$ for $-10 \le x \le 10$ and $-3 \le y \le 3$. *[1]*

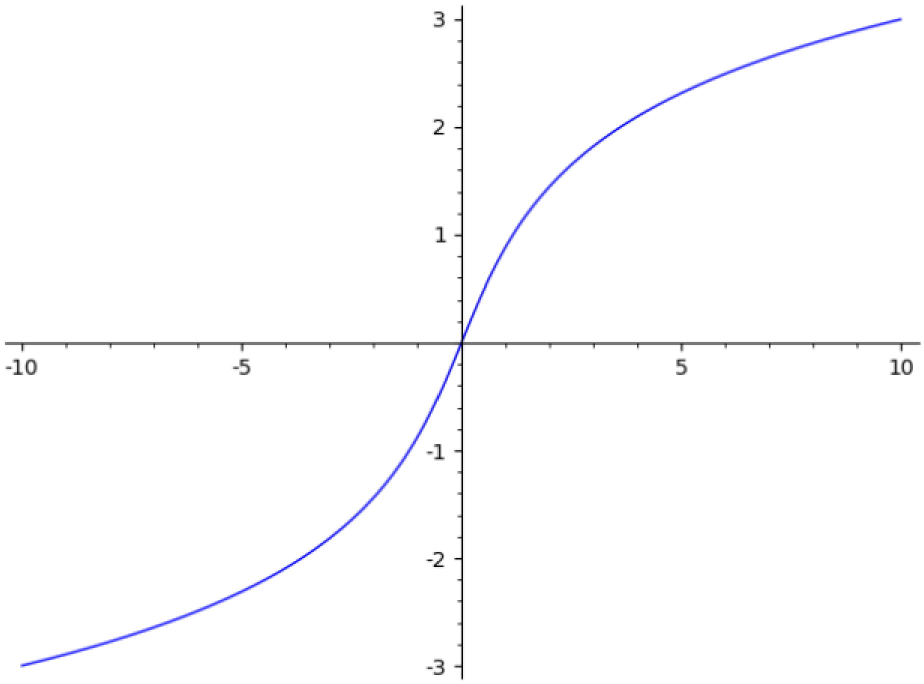
c. Use SageMath to plot $y = f^{-1}(x)$, otherwise known as $x = f(y)$, for $-3 \le x \le 3$ and $-10 \le y \le 10$. *[2]*

SOLUTIONS. **a.** $\ln(t)$ is defined for all $t > 0$. Note that $\sqrt{x^2+1} > \sqrt{x^2} = |x|$ for all real numbers x, from which it follows that $x + \sqrt{x^2+1} > 0$ for all x. Putting these together, $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ is defined for all x, *i.e.* its domain is $\mathbb{R} = (-\infty, \infty)$. $\square$

b. Here we go:

```
[1]: # 1b
     plot( log( x + sqrt(x^2 + 1) ), -10, 10 )
```

[1]:

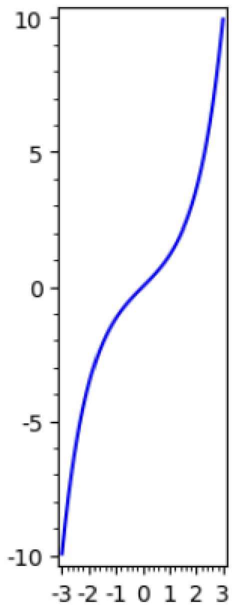


$\square$

c. Here we go again:

```
[2]: # 1c
     var('y')
     implicit_plot( x == log( y + sqrt( y^2 + 1 ) ), (x,-3,3), (y,-10,10) )
```

[2]:



■

2. a. Ask SageMath to solve the equation $x = f(y)$ for y. *[1]*

SageMath will probably give you a solution with a square root in which y occurs ...

b. Rearrange the solution SageMath gave you in part **a** to eliminate the square root and ask SageMath to solve the modified equation for y. *[2]*

This time SageMath should give you an answer for y in terms of x only.

c. What is $f^{-1}(x)$ in terms of x? What is the domain of $f^{-1}(x)$? *[1]*

SOLUTIONS. **a.** Here we go:

```
[3]: # 2a
     solve( x == log( y + sqrt( y^2 + 1 ) ), y )
```

```
[3]: [y == -sqrt(y^2 + 1) + e^x]
```

That is, SageMath lazily "solves" the equation it was given by isolating *a* y, returning $y = -\sqrt{y^2+1} + e^x$, rather than giving us y in terms of x only. [Bad SageMath!] □

b. We take the output we got in part **a** and rearrange it to eliminate the square root:

$$y = -\sqrt{y^2+1} + e^x \implies \sqrt{y^2+1} = e^x - y \implies y^2 + 1 = (e^x - y)^2$$

We now ask SageMath to pretty please solve this rearranged equation for y:

```
[4]: # 2b
     solve( y^2 + 1 == (y - e^x)^2, y )
```

```
[4]: [y == 1/2*(e^(2*x) - 1)*e^(-x)]
```

That is, SageMath tells us that $y = \frac{1}{2}\left(e^{2x} - 1\right) e^{-x}$. [OK, I accidentally typed in `y - e^x` instead of `e^x - y`, but since the expression is being squared, that doesn't really make a difference.] $\square$

c. Since we started with $x = f(y)$ and solved for y, the answer to part **b** tells us that $f^{-1}(x) = y = \frac{1}{2}\left(e^{2x} - 1\right) e^{-x}$.

As e^t is defined for all real numbers t, $f^{-1}(x)$ is defined for all x. The only thing that could go wrong is if $e^{-x} = \frac{1}{e^x}$ ended up dividing by 0, but this can't happen because $e^t > 0$ for all real t. Thus $f^{-1}(x)$ has domain $\mathbb{R} = (-\infty, \infty)$. $\blacksquare$

3. Work out what $f^{-1}(x)$ is in terms of x by hand, showing all the steps. *[2.5]*

Solution. Since $y = f^{-1}(x)$ exactly when $x = f(y)$, we solve the latter equation for y in terms of x.

$$\begin{aligned}
x = f(y) = \ln\left(y + \sqrt{y^2+1}\right) &\iff e^x = y + \sqrt{y^2+1} \\
&\iff e^x - y = \sqrt{y^2+1} \\
&\iff \left(e^x - y\right)^2 = y^2 + 1 \\
&\iff \left(e^x\right)^2 - 2ye^x + y^2 = y^2 + 1 \\
&\iff \left(e^x\right)^2 - 2ye^x = 1 \\
&\iff \left(e^x\right)^2 - 1 = 2ye^x \\
&\iff y = \frac{\left(e^x\right)^2 - 1}{2e^x}
\end{aligned}$$

This can be rearranged, using the fact that $\frac{1}{e^x} = e^{-x}$, into $\frac{1}{2}\left(\left(e^x\right)^2 - 1\right) e^{-x}$, which is the answer given by SageMath in the solution to **2b** above. Humans inclined to make things look neat would probably rearrange it until they get something like $y = \frac{e^x - e^{-x}}{2}$.

Thus $f^{-1}(x) = \frac{\left(e^x\right)^2 - 1}{2e^x} = \frac{1}{2}\left(\left(e^x\right)^2 - 1\right) e^{-x} = \frac{e^x - e^{-x}}{2}$, depending on taste. $\blacksquare$

Note. $f^{-1}(x)$ is commonly known as the hyperbolic function $\sinh(x) = \frac{e^x - e^{-x}}{2}$ (usually pronounced as "sinch"), and $f(x)$ is its inverse, $\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$.