Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Solutions to Quiz #11
More Integration

Please show all your work when answering the questions below. Do them by hand, please! Feel free to check your work using SageMath, though.

1. Work out $\displaystyle\int \sinh(x)\cos(x)\,dx$. *[2.5]*

NOTE: Recall that $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. These two functions are each other's derivatives, and hence also each other's antiderivatives. That is a $\cos(x)$ in the integral, though, not a $\cosh(x)$.

SOLUTION. We will use integration by parts twice, and do a little algebra after that.

$$\begin{aligned}
\int \sinh(x)\cos(x)\,dx &= \sinh(x)\sin(x) - \int \cosh(x)\sin(x)\,dx \\
&\qquad \text{where} \quad \begin{array}{ll} u = \sinh(x) & v' = \cos(x) \\ u' = \cosh(x) & v = \sin(x) \end{array} \\
&= \sinh(x)\sin(x) - \left[\cosh(x)\left(-\cos(x)\right) - \int \sinh(x)\left(-\cos(x)\right)\,dx\right] \\
&\qquad \text{where} \quad \begin{array}{ll} s = \cosh(x) & t' = \sin(x) \\ s' = \sinh(x) & t = -\cos(x) \end{array} \\
&= \sinh(x)\sin(x) + \cosh(x)\cos(x) - \int \sinh(x)\cos(x)\,dx
\end{aligned}$$

We now solve for $\displaystyle\int \sinh(x)\cos(x)\,dx$.

$$\begin{aligned}
2\int \sinh(x)\cos(x)\,dx &= \sinh(x)\sin(x) + \cosh(x)\cos(x) \\
\implies \int \sinh(x)\cos(x)\,dx &= \frac{\sinh(x)\sin(x) + \cosh(x)\cos(x)}{2} + C \qquad \square
\end{aligned}$$

CHECK:

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In [1]: integral(sinh(x)*cos(x),x)
Out[1]: 1/4*((e^(2*x) + 1)*cos(x) + (e^(2*x) - 1)*sin(x))*e^(-x)
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Note that SageMath puts the answer in terms of e^x, *i.e.* in terms of the definitions of $\sinh(x)$ and $\cosh(x)$. The answers are actually the same after some algebra.

2. Compute $\displaystyle\int_0^1 x\arctan(x)\,dx$. *[2.5]*

NOTE: Just in case, $\arctan(0)=0$ and $\arctan(1)=\dfrac{\pi}{4}$.

SOLUTION. We will use little algebraic trick after using integration by parts.

$$\int_0^1 x\arctan(x)\,dx = \frac{x^2}{2}\arctan(x)\Big|_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$\text{where}\quad \begin{array}{ll} u=\arctan(x) & v'=x \\ u'=1/\left(1+x^2\right) & v=x^2/2 \end{array}$$

$$= \frac{x^2}{2}\arctan(x)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\arctan(x)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\arctan(x)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{1+x^2}{1+x^2}\,dx - \frac{1}{2}\int_0^1 \frac{-1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\arctan(x)\Big|_0^1 - \frac{1}{2}\int_0^1 1\,dx + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\arctan(x)\Big|_0^1 - \frac{x}{2}\Big|_0^1 + \frac{1}{2}\arctan(x)\Big|_0^1$$

$$= \left[\frac{1^2}{2}\arctan(1) - \frac{0^2}{2}\arctan(0)\right] - \left[\frac{1}{2}-\frac{0}{2}\right]$$

$$+ \left[\frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(0)\right]$$

$$= \left[\frac{1}{2}\cdot\frac{\pi}{4} - 0\right] - \left[\frac{1}{2}-0\right] + \left[\frac{1}{2}\cdot\frac{\pi}{4}-0\right]$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2} \approx 0.2854$$

CHECK:

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In [2]: integral(x*arctan(x),x,0,1)
Out[2]: 1/4*pi - 1/2
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