Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Solutions to the Final Examination
19:00-22:00 in ENW 114 on Wednesday, 14 June.

Instructions: Do both of parts **I** and **II**, and, if you wish, part **III**. Please show all your work, justify all your answers, and simplify these where you reasonably can. When you are asked to do k of n questions, only the first k that are not crossed out will be marked. *If you have a question, or are in doubt about something,* **ask!**

Aids: Any calculator, as long as it can't communicate with other devices; (all sides of) one letter- or A4-size sheet; one natural intelligence.

Part I. Do all four (4) of **1–4**.

1. Compute $\frac{dy}{dx}$ as best you can in any four (4) of **a–f**. *[20 = 4 × 5 each]*

a. $y = x\tan(x)$ **b.** $y = \frac{\cos(x)}{x}$ **c.** $y = \int_1^{x/2} \cos(t)\,dt$

d. $y = (x-3)^{10}$ **e.** $y = \ln(1+e^x)$ **f.** $y = \sin^2(\ln(x))$

SOLUTIONS. **a.** *Product Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}(x\tan(x)) = \left[\frac{d}{dx}x\right]\tan(x) + x\left[\frac{d}{dx}\tan(x)\right] = 1\tan(x) + x\sec^2 x$$
$$= \tan(x) + x\sec^2(x) \quad \square$$

b. *Quotient Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos(x)}{x}\right) = \frac{\left[\frac{d}{dx}\cos(x)\right]x - \cos(x)\left[\frac{d}{dx}x\right]}{x^2} = \frac{-\sin(x)\cdot x - \cos(x)\cdot 1}{x^2}$$
$$= -\frac{x\sin(x)+\cos(x)}{x^2} \quad \square$$

c. *Fundamental Theorem of Calculus and Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_1^{x/2}\cos(t)\,dt\right) = \frac{d}{du}\left(\int_1^{u}\cos(t)\,dt\right)\cdot\frac{du}{dx} \quad \text{where } u = \frac{x}{2}$$
$$= \cos(u)\cdot\frac{d}{dx}\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)\cdot\frac{1}{2} = \frac{1}{2}\cos\left(\frac{x}{2}\right) \quad \square$$

c. *Integration and Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_1^{x/2}\cos(t)\,dt\right) = \frac{d}{dx}\left(\sin(t)\Big|_1^{x/2}\right) = \frac{d}{dx}\left(\sin\left(\frac{x}{2}\right) - \sin(1)\right)$$
$$= \cos\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right) - 0 = \cos\left(\frac{x}{2}\right)\cdot\frac{1}{2} = \frac{1}{2}\cos\left(\frac{x}{2}\right) \quad \square$$

d. *Power Rule and Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}(x-3)^{10} = 10(x-3)^9 \cdot \frac{d}{dx}(x-3) = 10(x-3)^9 \cdot (1-0) = 10(x-3)^9 \quad \square$$

e. *Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(1+e^x\right) = \frac{1}{1+e^x} \cdot \frac{d}{dx}\left(1+e^x\right) = \frac{1}{1+e^x} \cdot \left(0+e^x\right) = \frac{e^x}{1+e^x} \quad \square$$

f. *The Chain Rule Rules, with a side of the Power Rule and a trigonometric identity.*

$$\frac{dy}{dx} = \frac{d}{dx}\sin^2\left(\ln(x)\right) = 2\sin\left(\ln(x)\right) \cdot \frac{d}{dx}\sin\left(\ln(x)\right) = 2\sin\left(\ln(x)\right) \cdot \cos\left(\ln(x)\right) \cdot \frac{d}{dx}\ln(x)$$
$$= 2\sin\left(\ln(x)\right)\cos\left(\ln(x)\right) \cdot \frac{1}{x} = \frac{\sin\left(2\ln(x)\right)}{x} \quad \blacksquare$$

2. Evaluate any four (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a.} \int \frac{x}{x^2+1}\,dx \qquad \textbf{b.} \int_0^{e-1} \frac{x}{x+1}\,dx \qquad \textbf{c.} \int_0^{\pi} x\cos(x)\,dx$$

$$\textbf{d.} \int \frac{x^2+x}{x+1}\,dx \qquad \textbf{e.} \int \tan^2(x)\,dx \qquad \textbf{f.} \int_0^1 2x^3e^{x^2}\,dx$$

SOLUTIONS. **a.** *Substitution Rule.* We will use the substitution $u = x^2+1$, so $\frac{du}{dx} = 2x$, and thus $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$.

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{u} \cdot \frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln(u) + C = \frac{1}{2}\ln\left(x^2+1\right) + C \quad \square$$

b. *Substitution Rule.* We will use the substitution $u = x+1$, so $x = u-1$ and $\frac{du}{dx} = 1$, and thus $dx = du$. We will also change the limits as we go along: $\begin{matrix} x & 0 & e-1 \\ u & 1 & e \end{matrix}$

$$\int_0^{e-1} \frac{x}{x+1}\,dx = \int_1^e \frac{u-1}{u}\,du = \int_1^e \left(1-\frac{1}{u}\right)du = \left(u-\ln(u)\right)\Big|_1^e$$
$$= (e-\ln(e)) - (1-\ln(1)) = (e-1)-(1-0) = e-2 \quad \square$$

c. *Integration by parts.* We will use the parts $u = x$ and $v' = \cos(x)$, so $u' = 1$ and $v = \sin(x)$.

$$\int_0^{\pi} x\cos(x)\,dx = x\sin(x)\Big|_0^{\pi} - \int_0^{\pi} 1\sin(x)\,dx = \pi\sin(\pi) - 0\sin(0) - \left(-\cos(x)\right)\Big|_0^{\pi}$$
$$= \pi\cdot 0 - 0 + \left(\cos(x)\right)\Big|_0^{\pi} = 0 + \cos(\pi) - \cos(0) = (-1) - 1 = -2 \quad \square$$

d. *Algebra and the Power Rule.* Observe:

$$\int \frac{x^2+x}{x+1}\,dx = \int \frac{x(x+1)}{x+1}\,dx = \int x\,dx = \frac{x^2}{2} + C \quad \square$$

e. *A trigonometric identity and a bit of the Power Rule.* Observe:

$$\int \tan^2(x)\,dx = \int \left(\sec^2(x) - 1\right)\,dx = \tan(x) - x + C \quad \square$$

f. *Substitution and integration by parts.* We will first use the substitution $w = x^2$, so $\dfrac{dw}{dx} = 2z$, and thus $dw = 2x\,dx$, and change the limits as we go: $\begin{matrix} x & 0 & 1 \\ w & 0 & 1 \end{matrix}$

$$\int_0^1 2x^3 e^{x^2}\,dx = \int x^2 e^{x^2}\,2x\,dx = \int_0^1 we^w\,dw$$

We now apply integration by parts, with $u = w$ and $v' = e^w$, so $u' = 1$ and $v = e^w$.

$$\begin{aligned}\int_0^1 2x^3 e^{x^2}\,dx &= \int_0^1 we^w\,dw = we^w\big|_0^1 - \int_0^1 1e^w\,dw = 1e^1 - 0e^0 - e^w\big|_0^1 \\ &= e - 0 - \left(e^1 - e^0\right) = e - e + 1 = 1 \quad \blacksquare\end{aligned}$$

3. Do any four (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Compute $\displaystyle\lim_{x\to 0} \frac{x}{\tan(x)}$.

b. Use the ε–δ definition of limits to verify that $\displaystyle\lim_{x\to 2} (4x-7) = 1$.

c. At what point (x, y) does the graph of $y = e^x$ have a tangent line with slope 2?

d. Sketch the region between $y = x + 2$ and $y = x^2$, for $-1 \le x \le 2$, and find its area.

e. Let $f(x) = \begin{cases} x\ln(x) & x > 0 \\ 0 & x \le 0 \end{cases}$. Determine whether $f(x)$ is continuous at $x = 0$.

f. Suppose $f'(x) = x^2$ and $f(1) = 1$. What is the function $f(x)$?

Solutions. **a.** We will use l'Hôpital's Rule to help us out, which are permitted to do because numerator $x \to 0$ and denominator $\tan(x) \to 0$ as $x \to 0$.

$$\lim_{x\to 0} \frac{x}{\tan(x)} = \lim_{x\to 0} \frac{\frac{d}{dx}x}{\frac{d}{dx}\tan(x)} = \lim_{x\to 0} \frac{1}{\sec^2(x)} = \frac{1}{\sec^2(0)} = \frac{1}{1^2} = 1 \quad \square$$

b. We need to check that for any $\varepsilon > 0$ one can find a $\delta > 0$ such that if $|x-2| < \delta$, then $|(4x-7)-1| < \varepsilon$. As is common in such cases, we will tryt to reverse-engineer the required δ from $|(4x-7)-1| < \varepsilon$.

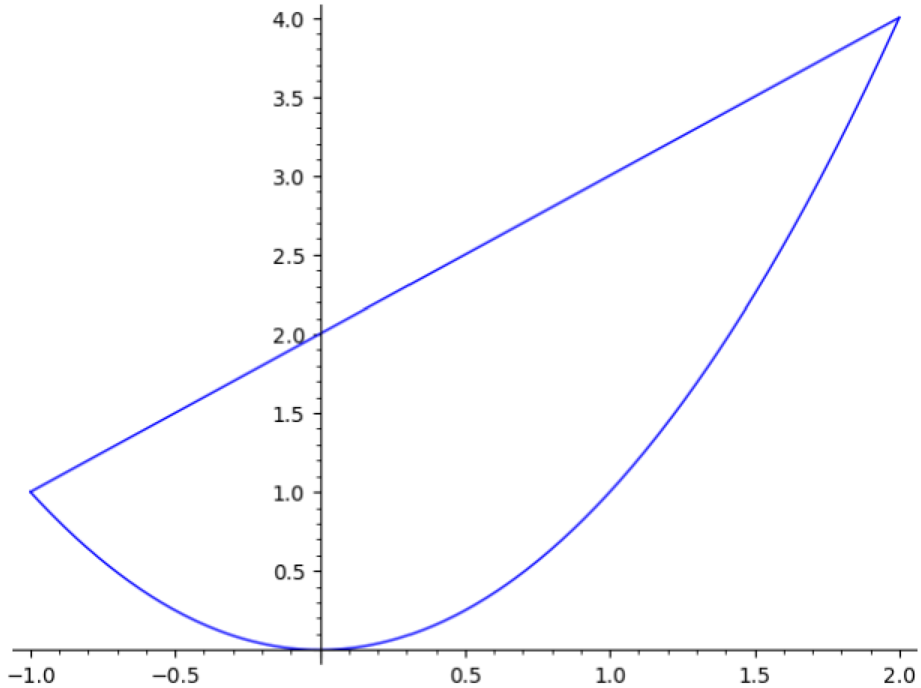
$$\begin{aligned}|(4x-7)-1| < \varepsilon &\iff |4x-8| < \varepsilon \iff |4(x-2)| < \varepsilon \\ &\iff 4|x-2| < \varepsilon \iff |x-2| < \frac{\varepsilon}{4}\end{aligned}$$

Observe that every step in this process is reversible. It follows that if we are given any $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{4}$, then it will be true that if $|x-2| < \delta$, then $|(4x-7)-1| < \varepsilon$. Thus $\lim_{x \to 2} (4x-7) = 1$ by the ε–δ definition of limits. □

c. Since $\frac{dy}{dx}$ gives the slope of the tangent line to $y = e^x$ at any point at which it is defined, we need to solve the equation $2 = \frac{dy}{dx} = \frac{d}{dx} e^x = e^x$. By the definition of the logarithm function as the inverse function to e^x, $e^x = 2 \iff x = \ln(2)$. The corresponding y-value is [imagine a dramatic drum roll] $y = e^{\ln(2)} = 2$. It follows that the graph of $y = e^x$ has slope 2 at the point $(x, y) = (\ln(2), 2)$. □

d. Cheating – just a bit! – Here is a sketch of the curves, as drawn by SageMath:

```
In [1]: plot(x+2,-1,2) + plot(x^2,-1,2)
Out[1]:
```



It's pretty clear from the plot that for x between -1 and 2, $y = x - 1$ is indeed above $y = x^2$. It follows that the area of the region between the curves is:

$$\begin{aligned}
\int_{-1}^{2} \left(x + 2 - x^2\right)\, dx &= \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Bigg|_{-1}^{2} \\
&= \left(\frac{2^2}{2} + 2 \cdot 2 - \frac{2^3}{3}\right) - \left(\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right) \\
&= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 8 - \frac{9}{3} - \frac{1}{2} = \frac{9}{2} \quad \square
\end{aligned}$$

e. By definition, $f(x) = \begin{cases} x\ln(x) & x > 0 \\ 0 & x \leq 0 \end{cases}$ is continuous at $x = 0$ if it defined there – which it is, with $f(0) = 0$ – and $\lim_{x \to 0} f(x) = f(0) = 0$. It's pretty obvious that the the limit as x approaches 0 from the left works: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$. It remains to check that the limit as x approaches 0 from the right also works.

We will apply l'Hôpital's Rule after a little algebraic manipulation and a quick check that we *can* legitimately apply it.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x\ln(x) = \lim_{x \to 0^+} \frac{\ln(x) \to +\infty}{\frac{1}{x} \quad \to +\infty} = \lim_{x \to 0^+} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$
$$= \lim_{x \to 0^+} \frac{1}{x} \cdot \left(-\frac{x^2}{1}\right) = \lim_{x \to 0^+} (-x) = 0$$

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$, we have that $\lim_{x \to 0} f(x) = f(0) = 0$, so $f(x)$ is indeed continuous at $x = 0$. □

f. We are given that $f'(x) = x^2$ and $f(1) = 1$. The former and the Fundamental Theorem of Calculus imply, with the help of the Power Rule, that

$$f(x) = \int f'(x)\,dx = \int x^2\,dx = \frac{x^3}{3} + C$$

for some unknown constant C. The second fact that we are given, that $f(1) = 1$, lets us pin down C:

$$\frac{1^3}{3} + C = f(1) = 1 \implies C = 1 - \frac{1}{3} = \frac{2}{3}$$

Thus $f(x) = \dfrac{x^3}{3} + \dfrac{2}{3} = \dfrac{x^3 + 2}{3}$. ■

4. Find the domain, intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, maximum and minimum points, intervals of concavity, and inflection points of $f(x) = \dfrac{x}{1 + x^2}$. *[15]*

SOLUTION. We run through the indicated checklist:

i. Domain. Since $1 + x^2 \geq 1 > 0$ for all x, the denominator of $f(x) = \dfrac{x}{1 + x^2}$ is never 0. Since $f(x)$ is a ratio of polynomials which are defined, continuous, and differentiable everywhere, and the denominator is never 0, $f(x)$ is also defined, continuous, and differentiable for all x.

ii. Intercepts. $f(0) = \dfrac{0}{1 + 0^2} = 0$, so the y-intercept is 0, and is also an x-intercept. Since $\dfrac{x}{1 + x^2} = 0$ exactly when the numerator, *i.e.* x, is 0, $(0, 0)$ is also the only x-intercept.

iii. Vertical and Horizontal Asymptotes. As was noted above, $f(x)$ is defined and continuous for all x, so it cannot have any vertical asymptotes, which are a type of discontinuity.

It remains to check for horizontal asymptotes, which we do with a little help from l'Hôpital's Rule, being careful to check that we can legitimately apply it:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x \quad \to -\infty}{1+x^2 \to +\infty} = \lim_{x\to-\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}\left(1+x^2\right)} = \lim_{x\to-\infty} \frac{1 \quad \to 1}{2x \to -\infty} = 0^-$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x \quad \to +\infty}{1+x^2 \to +\infty} = \lim_{x\to+\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}\left(1+x^2\right)} = \lim_{x\to+\infty} \frac{1 \quad \to 1}{2x \to +\infty} = 0^+$$

Thus $y = 0$ is a horizontal asymptote for $y = f(x)$ in both directions, which $f(x)$ approches from below as $x \to -\infty$ and from above as $x \to +\infty$.

iv. Increase/decrease and max/min. We first need to compute the derivative, which we do with the help of the Quotient and Power Rules.

$$f'(x) = \frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{\left[\frac{d}{dx}x\right]\left(1+x^2\right) - x\left[\frac{d}{dx}\left(1+x^2\right)\right]}{\left(1+x^2\right)^2}$$

$$= \frac{1\left(1+x^2\right) - x\cdot 2x}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2}$$

Observe that the denominator of the derivative, $\left(1+x^2\right)^2$, is always positive; in fact, it is always ≥ 1. It follows that the derivative is defined for all x, and is positive, negative, or 0, exactly as the numerator is positive, negative, or 0, respectively. Thus:

$$f'(x) \begin{matrix}<\\=\\>\end{matrix} 0 \iff 1-x^2 \begin{matrix}<\\=\\>\end{matrix} 0 \iff 1 \begin{matrix}<\\=\\>\end{matrix} x^2 \iff \begin{matrix}x<-1 \text{ or } x>1\\ x=\pm 1\\ -1<x<1\end{matrix}$$

It follows that $f(x)$ is decreasing when $x < -1$ or $x > 1$, and increasing when $-1 < x < 1$, with critical points at $x = -1$ and $x = 1$. Since $f(x)$ is decreasing before and increasing after $x = -1$, the point $(-1, f(-1)) = \left(-1, -\frac{1}{2}\right)$ is a minimum point, and since $f(x)$ is increasing before and decreasing after $x = 1$, the point $(1, f(1)) = \left(1, \frac{1}{2}\right)$ is a maximum point. We summarize all this in a table:

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\downarrow$	$-\frac{1}{2}$	$\uparrow$	$\frac{1}{2}$	$\downarrow$
		min		max	

v. Concavity and inflection. We first need to compute the second derivative, which we do with the help of the Quotient and Power Rules.

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{1-x^2}{\left(1+x^2\right)^2}\right) = \frac{\left[\frac{d}{dx}\left(1-x^2\right)\right]\left(1+x^2\right)^2 - \left(1-x^2\right)\left[\frac{d}{dx}\left(1+x^2\right)^2\right]}{\left(\left(1+x^2\right)^2\right)^2}$$

$$= \frac{-2x\left(1+x^2\right)^2 - \left(1-x^2\right)2\left(1+x^2\right)2x}{\left(1+x^2\right)^4} = \frac{-2x\left(1+x^2\right) - 4x\left(1-x^2\right)}{\left(1+x^2\right)^3}$$

$$= \frac{-2x-2x^3-4x+4x^3}{\left(1+x^2\right)^3} = \frac{2x^3-6x}{\left(1+x^2\right)^3} = \frac{2x\left(x^2-3\right)}{\left(1+x^2\right)^3}$$

Observe that the denominator of the second derivative, $\left(1+x^2\right)^3$, is always positive because $1+x^2 \geq 1$ for all x. It follows that the second derivative is defined for all x, and is positive, negative, or 0, exactly as the numerator is positive, negative, or 0, respectively. Thus:

$$f''(x) \begin{matrix} < \\ = 0 \\ > \end{matrix} \iff 2x\left(x^2-3\right) \begin{matrix} < \\ = 0 \\ > \end{matrix} \iff \begin{matrix} x<0 \text{ and } x^2>3, \text{ or } x>0 \text{ and } x^2<3 \\ x=0, \text{ or } x^2=3 \\ x<0 \text{ and } x^2<3, \text{ or } x>0 \text{ and } x^2>3 \end{matrix}$$

$$\iff \begin{matrix} x<0 \text{ and } x<-\sqrt{3}, \text{ or } x<0 \text{ and } x>\sqrt{3}, \text{ or } x>0 \text{ and } \sqrt{3}<x<\sqrt{3} \\ x=0, \text{ or } x=-\sqrt{3}, \text{ or } x=\sqrt{3} \\ x<0 \text{ and } -\sqrt{3}<x<\sqrt{3}, \text{ or } x>0 \text{ and } x<-\sqrt{3}, \text{ or } x>0 \text{ and } x>\sqrt{3} \end{matrix}$$

$$\iff \begin{matrix} x<-\sqrt{3}, \text{ or } 0<x<\sqrt{3} \\ x=0, \text{ or } x=-\sqrt{3}, \text{ or } x=\sqrt{3} \\ -\sqrt{3}<x<0, \text{ or } x>\sqrt{3} \end{matrix} \qquad \begin{matrix} \text{After eliminating all the} \\ \text{impossible combinations} \\ \text{and the redundancies.} \end{matrix}$$

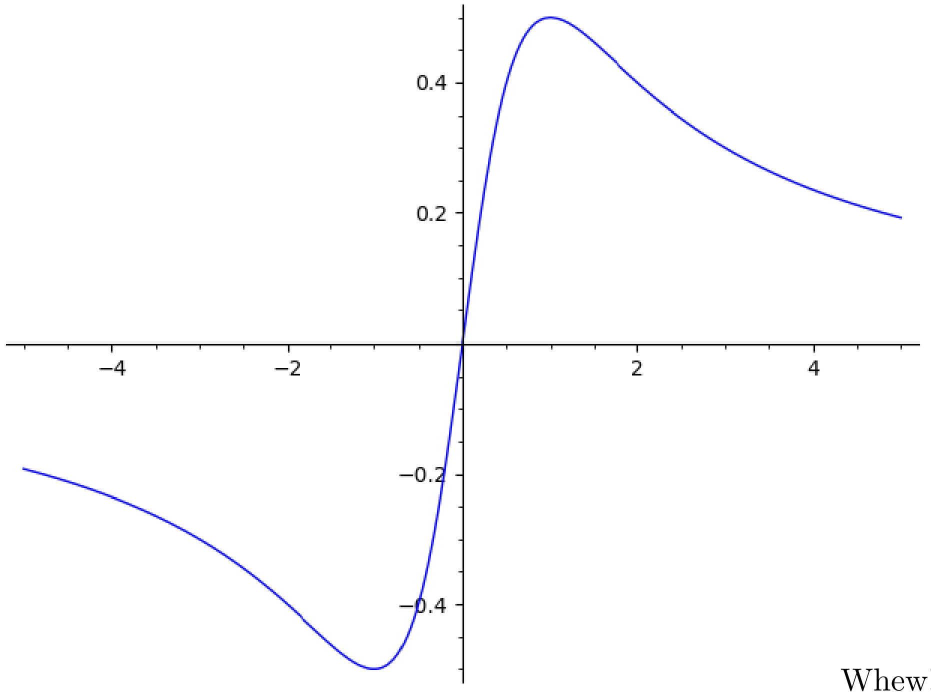
It follows that $f(x)$ is concave down when $x<-\sqrt{3}$ and when $0<x<\sqrt{3}$, and concave up when $-\sqrt{3}<x<0$ and when $x>\sqrt{3}$. This also means that we have inflection points at $x=-\sqrt{3}$, $x=0$, and $x=\sqrt{3}$; that is, $\left(-\sqrt{3},-\frac{\sqrt{3}}{4}\right)$, $(0,0)$, and $\left(\sqrt{3},\frac{\sqrt{3}}{4}\right)$ are the inflection points of $y=f(x)$. We summarize all this in a table:

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\frown$	$-\frac{\sqrt{3}}{4}$ infl	$\smile$	0 infl	$\frown$	$\frac{\sqrt{3}}{4}$ infl	$\smile$

vi. *Graph.* Cheating again, here is a plot of $y=\dfrac{x}{1+x^2}$ generated by SageMath:

```
In [1]: plot(x/(1+x^2),-5,5)
```

Out[1]:

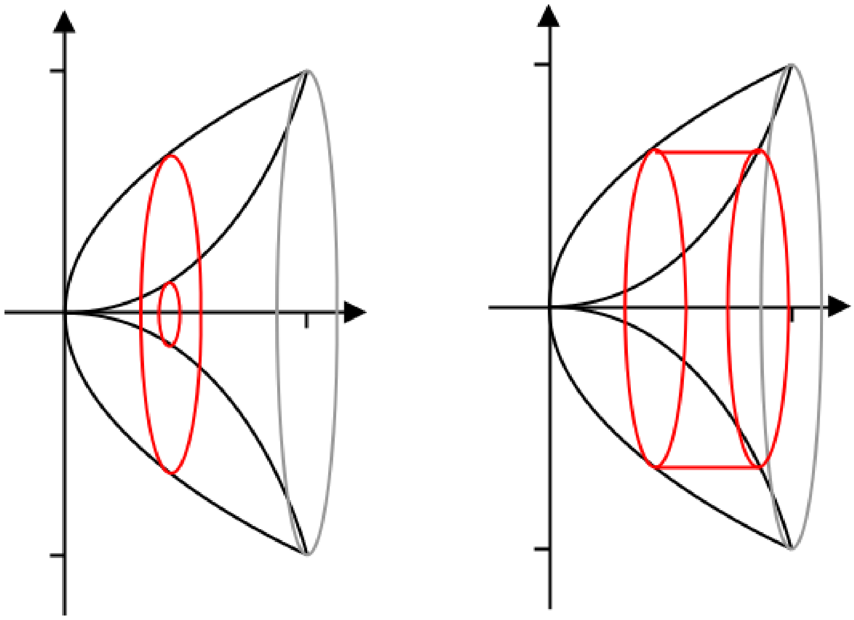


Whew! ∎

Part II. Do one (1) of **5**–**7**.

5. The region between $y = \sqrt{x}$ and $y = x^2$, for $0 \le x \le 1$, is revolved about the x-axis. Find the volume of the resulting solid. *[10]*

SOLUTION. We will find the volume using the disk/washer method and then also the cylindrical shell method. Here are two sketches of the solid, one with a generic washer drawn in and one with a generic cylindrical shell drawn in:



i. Disk/washer method. Since we revolved about the x-axis, the washers are perpendicular to it, so we will use x as our variable.

Observe that for x with $0 \le x \le 1$, $x^2 \le \sqrt{x}$. The washer at x, for some x between 0 and 1, has outer radius $R = y - 0 = \sqrt{x} - 0 = \sqrt{x}$ and inner radius $r = y - 0 = x^2 - 0 = x^2$. It follows that this washer has area $A(x) = \pi\left(R^2 - r^2\right) = \pi\left(\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right) = \pi\left(x - x^4\right)$. Thus the volume of the solid is given by:

$$\begin{aligned} V &= \int_0^1 A(x)\,dx = \int_0^1 \pi\left(x - x^4\right)\,dx = \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1 \\ &= \pi\left(\frac{1^2}{2} - \frac{1^5}{5}\right) - \pi\left(\frac{0^2}{2} - \frac{0^5}{5}\right) = \pi\left(\frac{1}{2} - \frac{1}{5}\right) - \pi\cdot 0 \\ &= \pi\left(\frac{5}{10} - \frac{2}{10}\right) - 0 = \frac{3\pi}{10} \end{aligned}$$

ii. Cylindrical shell method. Since we revolved about the x-axis, the cylindrical shells are parallel to the x-axis and perpendicular to the y-axis, so we will use y as our variable. Note that the region has the pleasant property that it has the same same range of y values as of x values, so $0 \le y \le 1$.

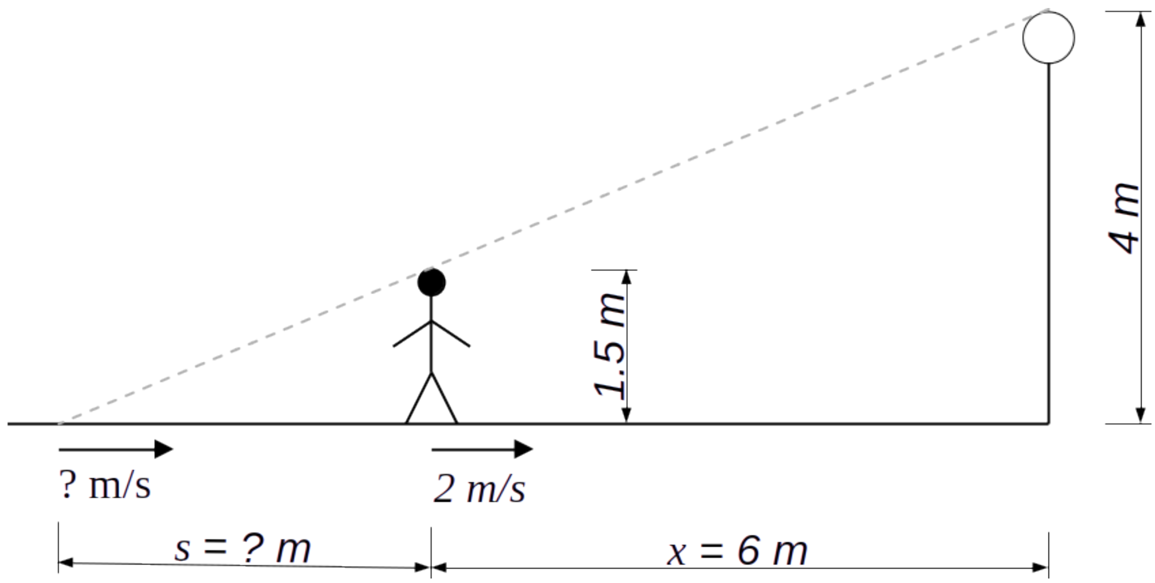
The cylindrical shell at y has radius $r = y - 0 = y$ and height (length?) $h = \sqrt{y} - y^2$ – the "upper" is from $y = x^2$ and the "lower" is from $y = \sqrt{x}$ – so it has area $A(y) =$

$2\pi rh = 2\pi y\left(\sqrt{y} - y^2\right) = 2\pi\left(y^{3/2} - y^3\right)$. Thus the volume of the solid is given by:

$$\begin{aligned} V &= \int_0^1 A(y)\,dy = \int_0^1 2\pi\left(y^{3/2} - y^3\right)\,dy = 2\pi\left(\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right)\Big|_0^1 \\ &= 2\pi\left(\frac{2\cdot 1^{5/2}}{5} - \frac{1^4}{4}\right) - 2\pi\left(\frac{2\cdot 0^{5/2}}{5} - \frac{0^4}{4}\right) = 2\pi\left(\frac{2}{5} - \frac{1}{4}\right) - 2\pi\cdot 0 \\ &= 2\pi\left(\frac{8}{20} - \frac{5}{20}\right) - 0 = 2\pi\cdot\frac{3}{20} = \frac{3\pi}{10} \end{aligned}$$

Fortunately for us, the two methods agree! :-) ■

6. Stick Figure, who is 1.5 m tall, walks at 2 m/s on level ground at night, straight towards a 4 m tall lit up lamppost. How fast is the tip of Stick's shadow moving along the ground at the instant that Stick is 6 m from the lamppost? *[10]*



Solution. Let x be the distance Stick Figure is from the lamppost and let s be the length of Stick's shadow, as in the annotated diagram above. We are given that $\frac{dx}{dt} = -2\ m/s$. (Since Stick is walking stright toward the lamppost, x must be decreasing ...)

From the setup, the triangle with the lamppost as one side and the tip of Stick's shadow as the opposite vertex is similar to the smaller triangle with Stick as one side and the tip of Stick's shadow as the opposite vertex. Corresponding sides of similar triangles must be in the same proportions, *e.g.* $\frac{1.5}{4} = \frac{s}{x+s}$. Cross-multiplying this equation gives $1.5(x+s) = 4s$, so $1.5x = 2.5s$, and thus $s = \frac{1.5}{2.5}x = \frac{3}{5}x$. The tip of Stick's shadow is therefore $x + s = x + \frac{3}{5}x = \frac{8}{5}x\ m$ from the lamppost at any given instant.

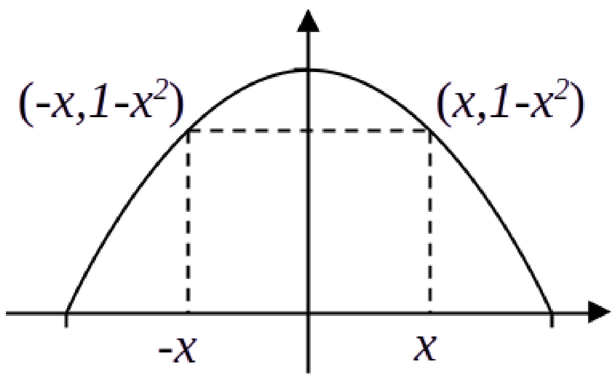
It now follows that Stick's shadow is moving along the ground at a rate given by:

$$\frac{d}{dt}(x+s) = \frac{d}{dt}\left(\frac{8}{5}x\right) = \frac{8}{5}\cdot\frac{dx}{dt} = \frac{8}{5}(-2) = -\frac{16}{5} = -3.2\ m/s$$

(Again, the sign is negative since the tip of the shadow is approaching the lamppost, so the distance is decreasing.) Since the question only asked for the speed (which notion doesn't care about direction) of the tip of Stick's shadow, 3.2 m/s is a valid answer; one work through the entire problem without that minus sign ... □

7. Find the maximum possible area of a rectangle whose corners are at $\left(x, 1-x^2\right)$, $\left(-x, 1-x^2\right)$, $(-x, 0)$, and $(x, 0)$, for some x with $0 \le x \le 1$. *[10]*

Solution. Here is a sketch of the setup:



As should be obvious from the sketch, such a rectangle has base $x - (-x) = 2x$ and height $1 - x^2$, and thus has area $A(x) = \text{base}\cdot\text{height} = 2x\left(1-x^2\right) = 2x - 2x^3$. Note that $A(0) = A(1) = 0$. Following the usual process for finding a maximum or minimum, we first look for critical points inside the given interval:

$$\begin{aligned} A'(x) &= \frac{d}{dx}\left(2x - 2x^3\right) = 2\cdot 1 - 2\cdot 3x^2 = 2\left(1-3x^2\right) \\ &= 0 \iff 1 - 3x^2 = 0 \iff x^2 = \frac{1}{3} \iff x = \pm\frac{1}{\sqrt{3}} \approx \pm 0.57735 \end{aligned}$$

Note that $A'(x)$ is defined and continuous in the interval $[0,1]$. The negative solution is not in the given interval $[0,1]$, but the positive one is. Evaluating $A(x)$ at $x = \frac{1}{\sqrt{3}}$ gives us:

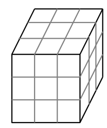
$$A\left(\frac{1}{\sqrt{3}}\right) = 2\cdot\frac{1}{\sqrt{3}} - 2\left(\frac{1}{\sqrt{3}}\right)^3 = \frac{2}{\sqrt{3}} - \frac{2}{3\sqrt{3}} = \frac{6}{3\sqrt{3}} - \frac{2}{3\sqrt{3}} = \frac{4}{3\sqrt{3}} \approx 0.76980$$

This must be a maximum because it is greater than $A(0) = A(1) = 0$ and there are no other critical points inside the given interval.

It follows that the maximum possible area of a rectangle with corners are at $\left(x, 1-x^2\right)$, $\left(-x, 1-x^2\right)$, $(-x, 0)$, and $(x, 0)$, for some x with $0 \le x \le 1$, is $\dfrac{4}{3\sqrt{3}} \approx 0.76980$. □

[Total = 85]

Part III. *Here be bonus points!* Do one or both of $\mathbf{2^3}$ and $\mathbf{3^2}$.



$\mathbf{2^3}$**.** A dangerously sharp tool is used to cut a cube with a side length of 3 *cm* into 27 smaller cubes with a side length of 1 *cm*. This can be done easily with six cuts. Can it be done with fewer? (Rearranging the pieces between cuts is allowed.) If so, explain how; if not, explain why not. *[1]*

SOLUTION. What's a solution? :-) □

$\mathbf{3^2}$**.** Write a haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

SOLUTION. What's a solution? :-) □

ENJOY THE REST OF THE SUMMER!