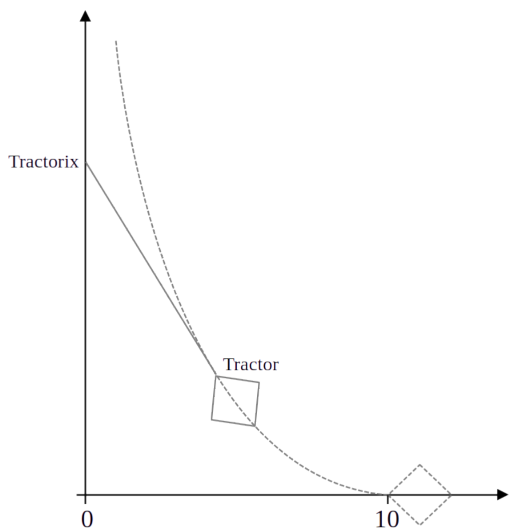
Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Summer 2023 (S61)

Solutions to Assignment #3
Tractor Pull

Before tackling this assignment, take a peek at the file `1110H-lab-20230517.pdf`, which you can find in the Labs folder in the Course Content section on Blackboard. Skimming and later referring to as necessary to Section 4.22 of Gregory Bard's book *Sage for Undergraduates* (in the SageMath folder in the Course Content section on Blackboard) is probably a good idea. *(Be warned that his book uses a format for declaring unknown functions that is not accepted by recent versions of SageMath. See the lab file above for how to do it right.)* If you wish to use another general purpose mathematics application, such as Maple or Mathematica, you may, but you're on your own for learning to use it and getting help.

In the beginning, Tractorix stands on the edge of a parking lot, holding onto one end of a 10 m cable whose other end is attached to a tractor. At this point the cable is stretched out at right angles to the edge of the parking lot. Tractorix begins to walk along the edge of the parking lot while holding onto the cable and towing the tractor. You may assume that the cable remains straight and taut and 10 m long, and that the path followed by the tractor has the property that the cable is always tangent to this path. (This is an example of a type of curve called a *tractrix*.) See the sketch below to help visualize all this.



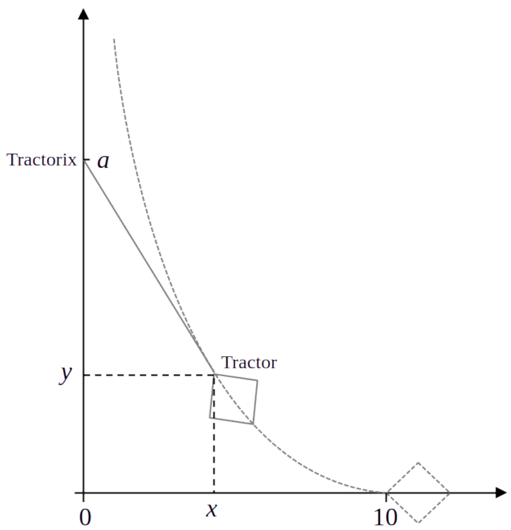
Your task will be to determine just what the path taken by the boat is. To help do this, we'll introduce Cartesian coordinates as suggested by the diagram. Let the y-axis run along the edge of the parking lot, with the origin at Tractorix's starting location, and with the direction Tractorix walks in being the positive direction, while the rope is initially stretched out along the positive x-axis. The path followed by the boat will be the graph of $y = f(x)$; note that $f(10) = 0$ and $f'(10) = 0$.

We will find the function $f(x)$ in two steps:

1. Find an expression for $\frac{dy}{dx} = f'(x)$ in terms of x. *[5]*

Hint: When the boat is at $(x, f(x))$, the rope is still 10 m long and its slope is $f'(x) = \frac{dy}{dx}$.

SOLUTION. Per the hint, suppose the tractor is at (x, y) and Tractorix is at $(0, a)$ at the same instant, as in the diagram below.



The points $(0, a)$, $(0, y)$, and (x, y) then form a right triangle, with the right angle at $(0, y)$ and the hypotenuse running between $(0, a)$ and (x, y). Since $(0, a)$ is the location of Tractorix and (x, y) is the location of the tractor at the same instant, we know from the setup that the rope coincides with the hypotenuse, which is therefore 10 m long and is part of the tangent line at (x, y) to the path followed by the tractor.

As the hypotenuse has length 10 and the other two sides of the triangle have lengths $a - y$ and x, the Pythagorean Theorem tells us that $x^2 + (a - y)^2 = 10^2 = 100$. It follows that $a - y = \sqrt{100 - x^2}$ – we take the positive root because $a - y > 0$ in this setup – and hence $y - a = -\sqrt{100 - x^2}$.

On the one hand, the slope of the tangent line, and hence the hypotenuse, is $f'(x) = \frac{dy}{dx}$ by the setup; on the other hand, the slope of the hypotenuse is given by $\frac{\Delta y}{\Delta x} = \frac{y - a}{x - 0} = \frac{y - a}{x}$. It follows that $f'(x) = \frac{dy}{dx} = \frac{y - a}{x}$.

Putting the conclusions in the above two paragraphs together allows us to eliminate the unknown a: $\frac{dy}{dx} = \frac{y - a}{x} = \frac{-\sqrt{100 - x^2}}{x}$. Thus if $y = f(x)$ is the path followed by the tractor as it is towed by Tractorix, then it satisfies the equation

$$\frac{dy}{dx} = \frac{-\sqrt{100 - x^2}}{x},$$

which is the desired expression of $\frac{dy}{dx}$ in terms of x. $\square$

2. Use SageMath to solve the differential equation you obtained in solving **1** for $y = f(x)$, where $f(10) = 0$ and $f'(10) = 0$. *[5]*

SOLUTION. We only need the fact that $f(10) = 0$ for our initial condition.

```
In [1]: y = function('y')(x)
        desolve( diff(y,x) == -sqrt(100-x^2)/x, y, ics=[10,0])

Out[1]: -sqrt(-x^2 + 100) - 10*log(20) + 10*log(20*(sqrt(-x^2 + 100) + 10)/abs(x))
```

That is, the path followed by the tractor is given by (the graph of) the function

$$f(x) = -\sqrt{100 - x^2} - 10\ln(20) + 10\ln\left(\frac{20\sqrt{100 - x^2} + 10}{|x|}\right).$$

This could be simplified and rearranged in various ways, but it's probably not worth the bother unless we know what we want to do with this next.

Feel free to experiment with adding $f'(10) = 0$ to the initial conditions, *i.e.* replace `ics=[10,0]` with `ics=[10,0,0]`, to see what happens. □