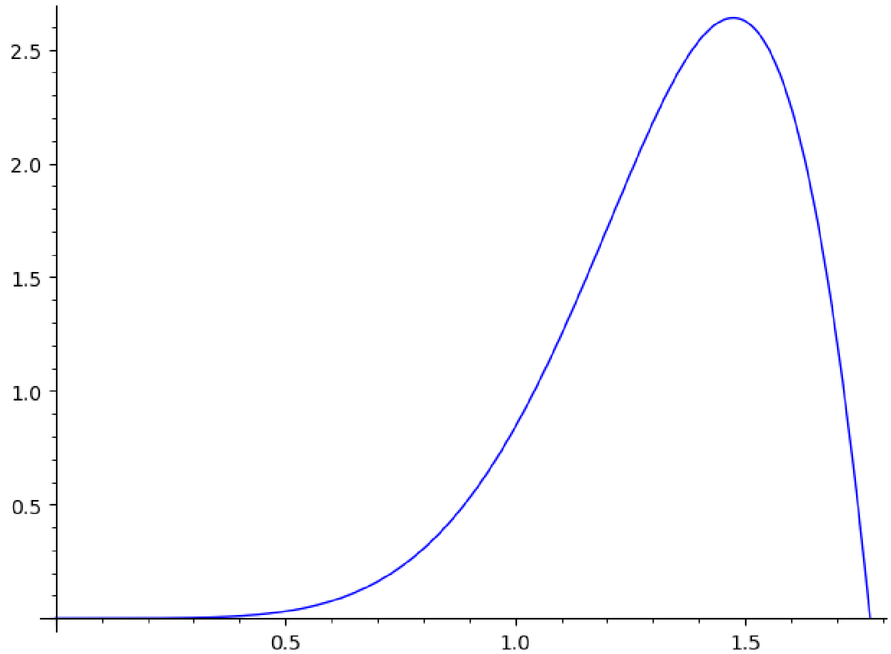
Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2021 (S62)

Solutions to Assignment #1
The easy way and the hard way ...

In this assignment you will attempt to compute the area of the region between the curve $y = x^3 \sin\left(x^2\right)$ for $0 \le x \le \sqrt{\pi}$, pictured below[1], and the x-axis.



As suggested by the title of this assignment, you will try to do this twice, with one method likely being easier than the other.

1. Find the antiderivative of $f(x) = x^3 \sin\left(x^2\right)$ and use it, along with the Fundamental Theorem of Calculus, to compute the area of the region given above. *[4]*

SOLUTION. We will find the antiderivative by computing the necessary indefinite integral using the substitution $w = x^2$, so $dw = 2x\,dx$ and thus $x\,dx = \frac{1}{2}\,dw$. This will bring us to an indefinite integral which we will use integration by parts on to proceed further.

$$\int x^3 \sin\left(x^2\right)\,dx = \int x^2 \sin\left(x^2\right) x\,dx = \int w \sin(w)\,\frac{1}{2} dw$$

Now use parts with $u = w$ and $v' = \sin(w)$, so $u' = 1$ and $v = -\cos(w)$.

$$= -\frac{1}{2} w\cos(w) - \frac{1}{2}\int 1\left(-\cos(w)\right)\,dw = -\frac{1}{2} w\cos(w) + \frac{1}{2}\int \cos(w)\,dw$$

$$= -\frac{1}{2} w\cos(w) + \frac{1}{2}\sin(w) + C = -\frac{1}{2} x^2 \cos\left(x^2\right) + \frac{1}{2}\sin\left(x^2\right) + C$$

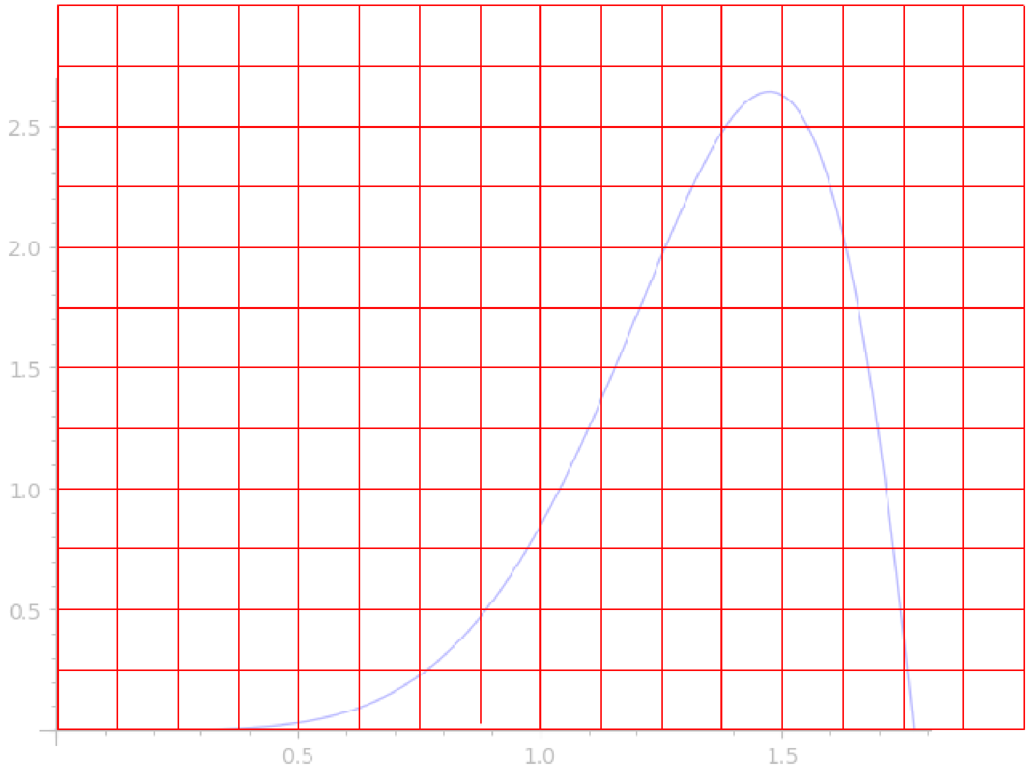
[1] This graph was generated by `SageMath` using the command: `plot(x^3*sin(x^2),0,sqrt(pi))`

Since the graph of $f(x) = x^3 \sin\left(x^2\right)$ is above the x-axis for all x with $0 \le x \le \sqrt{\pi}$, it follows that the definite integral $\displaystyle\int_0^{\sqrt{\pi}} x^3 \sin\left(x^2\right)\, dx$, which computes the weighted area between the graph and the x-axis actually gives the area of the region. By the Fundamental Theorem of Calculus, it follows that the area of the region is:

$$\begin{aligned}
\text{Area} &= \int_0^{\sqrt{\pi}} x^3 \sin\left(x^2\right)\, dx = \left[-\frac{1}{2}x^2\cos\left(x^2\right) + \frac{1}{2}\sin\left(x^2\right)\right]_0^{\sqrt{\pi}} \\
&= \left[-\frac{1}{2}\sqrt{\pi}^2\cos\left(\sqrt{\pi}^2\right) + \frac{1}{2}\sin\left(\sqrt{\pi}^2\right)\right] - \left[-\frac{1}{2}0^2\cos\left(0^2\right) + \frac{1}{2}\sin\left(0^2\right)\right] \\
&= \left[-\frac{1}{2}\pi\cos(\pi) + \frac{1}{2}\sin(\pi)\right] - \left[-\frac{1}{2}\cdot 0\cos(0) + \frac{1}{2}\sin(0)\right] \\
&= \left[-\frac{1}{2}\pi(-1) + \frac{1}{2}\cdot 0\right] - [0+0] = \frac{\pi}{2} \approx 1.5708 \qquad \blacksquare
\end{aligned}$$

2. Find the area of the region given above, as accurately as you can, entirely by hand[2] and *without* using the Fundamental Theorem of Calculus in any way. *[6]*

SOLUTION. *i. Counting points.* Consider the following diagram:



[2] With one exception: you may use a calculator to help with your arithmetic, up to and including computing (approximate) values of $f(x) = x^3 \sin\left(x^2\right)$.

The plot given in the assignment has been overlaid with a grid. The grid has 17 equally spaced vertical lines and 13 equally spaced horizontal lines, the outermost of which outline a rectangle 2 units wide and 3 units high which contains the entire region. (Note that the scaling in the plot is not the same for the vertical and horizontal axes.) The rectangle thus has area $2 \cdot 3 = 6$ and includes $17 \cdot 13 = 221$ intersection points where a vertical and a horizontal line of the grid meet. (This counts such points on the border of the rectangle.) A careful count yields 59 intersection points that are in the region. (Points on the region's border are counted, but points close to the border but not in the region are not counted.) That is, 59 out of a possible 221 grid intersection points are in the region, so $\frac{59}{221}$ is an approximate proportion of the area of the region to the area of the rectangle. Thus the area of the region is approximately $6 \cdot \frac{59}{221} \approx 1.608$.

This method got us to within about 0.03 of the correct value. Finer grids would be expected to give more accurate results. One interesting variation on this idea is to generate points in the rectangle randomly instead of using a grid to define them; such "Monte Carlo" methods can give very good results with relatively low computational costs in many applications. Another variation is to count the little boxes the grid divides the rectangles into, including those which are mostly in the region and excluding those that are mostly out of the region. This idea is similar to that used by Nicole Oresme (*c.* 1325–1382)[3] to estimate the area under a curve; his use of "latitude" and "longitude" as a coordinate system in contexts outside of geography anticipated the development of Cartesian coordinates a couple of centuries later. $\square$

ii. Rectangles using the right-hand rule. This time we'll compute the Riemann sum approximating the area using eight rectangles of equal width, with their heights determined by evaluating the function at the righthand endpoints of each interval. Since the width of the interval over which the region lives is $\sqrt{pi} - 0 = \sqrt{\pi} \approx 1.7725$, each rectangle will have width $\frac{\sqrt{\pi}}{8} \approx 0.4431$. Some (more) sloppy work with a calculator then gives us the following approximations:

rectangle	endpoint	f(endpoint)
1	0.2216	0.0005
2	0.4431	0.0170
3	0.6647	0.1256
4	0.8862	0.4921
5	1.1078	1.2801
6	1.3293	2.6472
7	1.5509	2.5051
8	1.7724	0.0011

[3] To quote the Wikipedia article about him, he "was a significant philosopher of the later Middle Ages. He wrote influential works on economics, mathematics, physics, astrology and astronomy, philosophy, and theology; was Bishop of Lisieux, a translator, a counselor of King Charles V of France, and one of the most original thinkers of 14th-century Europe."

It follows that the approximate area given by this sum of rectangles is:

$$\begin{aligned}\text{Area} &\approx (\text{common width}) \cdot (\text{sum of rectangles heights}) \\ &= 0.2216 \cdot (0.0005 + 0.0170 + 0.1256 + 0.4921 + 1.2801 + 2.6472 + 2.5051 + 0.0011) \\ &= 0.2216 \cdot 7.0687 = 1.5664\end{aligned}$$

This method got us within 0.005 of the correct value, albeit at the cost of punching a lot of keys on the calculator. (Imagine having to do this by looking up the sine values in a table and doing all the other work entirely by hand ...) Variations on this general idea would include using the middle points of the intervals to find the height of the rectangles, which often gives more accurate results, or by using the Trapezoid Rule or Simpson's Rule described in §8.6 of the text, which replace rectangles with more complicated shapes and usually give even more accurate results. ■