Mathematics 1120H – Calculus II: Integrals and Series

Trent University, Summer 2021 (S62)

Solutions to Quiz #~~5~~ 6

Tuesday, 27 July.

Do all of the following questions. Show all your reasoning in each solution. Please note that part marks are available in questions worth more than 0.5 points, so incomplete or incorrect solutions may still earn something.

1. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n x^{2n}}{4^n}$. *[2]*

Solution. As usual, we first use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)x^{2(n+1)}}{4^{n+1}}}{\frac{(-1)^n n x^{2n}}{4^n}} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)x^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n n x^{2n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)(n+1)x^2}{4n} \right| = \lim_{n\to\infty} \frac{x^2}{4} \cdot \frac{n+1}{n} \\
&= \frac{x^2}{4} \cdot \lim_{n\to\infty} \left(\frac{n}{n} + \frac{1}{n} \right) = \frac{x^2}{4} \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n} \right) \\
&= \frac{x^2}{4} \cdot (1+0) = \frac{x^2}{4}
\end{aligned}$$

It follows by the Ratio Test that the given series converges when $\dfrac{x^2}{4} < 1$, *i.e.* when $-2 < x < 2$, and diverges when $\dfrac{x^2}{4} > 1$, *i.e.* when $x < -2$ or $x > 2$. Thus the radius of convergence of this power series is $R = 2$.

To find the interval of convergence we also need to know what happens when $x = \pm 2$:

When $x = -2$, the given series is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n (-2)^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n n 4^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n n$. It diverges by the Divergence Test since $\displaystyle\lim_{n\to\infty} |(-1)^n n| = \lim_{n\to\infty} n = \infty \neq 0$.

Similarly, when $x = 2$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n 2^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n n 4^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n n$. It also diverges by the Divergence Test since $\displaystyle\lim_{n\to\infty} |(-1)^n n| = \lim_{n\to\infty} n = \infty \neq 0$.

Since the given series diverges when $x = \pm 2$, its interval of convergence is $(-2, 2)$. ■

Bonus. What function has this series as its Taylor series centred at $a = 0$? *[0.5]*

Solution. Here's the process I went through to figure this out. Apologies for the verbosity!

First, let's consolidate what we can in each term, since the x^{2n} appearing the nth term suggests that some sort of substitution may play a role:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n x^{2n}}{4^n} = \sum_{n=0}^{\infty} n \left(-\frac{x^2}{4} \right)^n$$

Second, following up on the idea of a substitution, let's write what we could consolidate as new variable, say $z = -\dfrac{x^2}{4}$, to simplify the problem. Thus:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n x^{2n}}{4^n} = \sum_{n=0}^{\infty} n\left(-\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty} n z^n$$

Third, we can simplify this a bit further by dropping the 0th term, since it has a factor of $n = 0$, giving us:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n x^{2n}}{4^n} = \sum_{n=0}^{\infty} n\left(-\frac{x^2}{4}\right)^n = \sum_{n=1}^{\infty} n z^n$$

On reflection, we could have done that at the very beginning ... Oh well, better late than never. :-)

Fourth, inspecting the last version of the series, it almost looks like the term-by-term derivative of the geometric series $\displaystyle\sum_{n=1}^{\infty} z^n$, except that the power of z is just a bit too big. we can fix that by pulling out one z from each term in the entire series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n x^{2n}}{4^n} = \sum_{n=0}^{\infty} n\left(-\frac{x^2}{4}\right)^n = \sum_{n=1}^{\infty} n z^n = z\sum_{n=1}^{\infty} n z^{n-1} = z\sum_{n=1}^{\infty} \frac{d}{dz} z^n$$

Fifth, since we know how to sum a geometric series when it converges, this means that:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n n x^{2n}}{4^n} &= \sum_{n=0}^{\infty} n\left(-\frac{x^2}{4}\right)^n = z\sum_{n=1}^{\infty} \frac{d}{dz} z^n = z\frac{d}{dz}\left(\sum_{n=1}^{\infty} z^n\right) = z\frac{d}{dz}\left(\frac{z}{1-z}\right) \\
&= z \cdot \frac{\left[\frac{d}{dz} z\right](1-z) - z\left[\frac{d}{dz}(1-z)\right]}{(1-z)^2} = z \cdot \frac{1(1-z) - z(-1)}{(1-z)^2} \\
&= z \cdot \frac{1-z+z}{(1-z)^2} = \frac{z}{(1-z)^2}
\end{aligned}$$

Sixth, we substitue back to put our new-found function in terms of x. Recall that $z = -\dfrac{x^2}{4}$. Thus:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n n x^{2n}}{4^n} &= \sum_{n=0}^{\infty} n\left(-\frac{x^2}{4}\right)^n = \frac{z}{(1-z)^2} = \frac{-\frac{x^2}{4}}{\left(1-\left(-\frac{x^2}{4}\right)\right)^2} \\
&= \frac{-x^2}{4\left(1+\frac{x^2}{4}\right)^2} = \frac{-x^2}{4 \cdot \frac{1}{16}\left(4+x^2\right)^2} = \frac{-x^2}{\frac{1}{4}\left(4+x^2\right)^2} = \frac{-4x^2}{\left(4+x^2\right)^2}
\end{aligned}$$

I think that's as nice as I can make it! Since a power series centred at a which is equal to a function must be the Taylor series of that function centred at a, it follows that the given series is the Taylor series centred at $a = 0$ of the function $f(x) = \dfrac{-4x^2}{(4+x^2)^2}$. ■

2. Let $f(x) = \dfrac{1}{2+x}$.

a. Use Taylor's formula to find the Taylor series centred at $a = -1$ of $f(x)$. *[2]*

b. Find the radius and interval of convergence of this Taylor series. *[1]*

SOLUTIONS. **a.** Taylor's formula tells us that the Taylor series of $f(x) = \dfrac{1}{2+x} = (2+x)^{-1}$ centred at $a = -1$ is the power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x-(-1))^n$, so what we really need to know is what the nth derivative of $f(x)$ at -1, *i.e.* $f^{(n)}(-1)$, is for each n:

n	$f^{(n)}(x)$	$f^{(n)}(-1)$
0	$(2+x)^{-1}$	1
1	$(-1)(2+x)^{-2}$	-1
2	$(-1)(-2)(2+x)^{-3}$	2
3	$(-1)(-2)(-3)(2+x)^{-4}$	-6
4	$(-1)(-2)(-3)(-4)(2+x)^{-5}$	24
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n n!(2+x)^{-(n+1)}$	$(-1)^n n!$
$\vdots$	$\vdots$	$\vdots$

It's not too hard to see the pattern developing above. Note that the formula for $f^{(n)}(-1)$ still works when $n = 0$ because $0! = 1$.

Thus the Taylor series of $f(x) = \dfrac{1}{2+x} = (2+x)^{-1}$ centred at $a = -1$ is

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(x-(-1))^n &= \sum_{n=0}^{\infty}(-1)^n(x+1)^n \\ &= 1-(x+1)+(x+1)^2-(x+1)^3+\cdots .\end{aligned}$$

We can check this by finding this Taylor series in a different way, by using the sum formula for a geometric series in reverse:

$$\begin{aligned}\frac{1}{2+x} = \frac{1}{1-(-(x+1))} &= 1-(x+1)+(x+1)^2-(x+1)^3+\cdots \\ &= 1-(x-(-1))+(x-(-1))^2-(x-(-1))^3+\cdots\end{aligned}$$

When a power series centred at a is equal to a function, it must be the Taylor series of that function centred at a. Since we got $f(x)$ equal to the power series obtained above, it follows that we got it right above. □

b. We could use the Ratio Test and such again, but there is an easier way if we recognize the Taylor series we obtained is a geometric series with common ratio $r = -(x-(-1)) = -(x+1)$ and first term 1. Such a series converges when $|r| < 1$ and diverges otherwise, so this Taylor series converges exactly when $|-(x-(-1))| = |x+1| < 1$. Unwinding the inequality, this means that the Taylor series we obtained converges when $-1 < x+1 < 1 \iff -2 < x < 0$, *i.e.* the interval of convergence of the series is $(-2, 0)$. The point about which the series is centred, $a = -1$, is the midpoint of this interval and is a distance of 1 from either endpoint, so the radius of convergence of this series is $R = 1$. ■

[Total = 5]