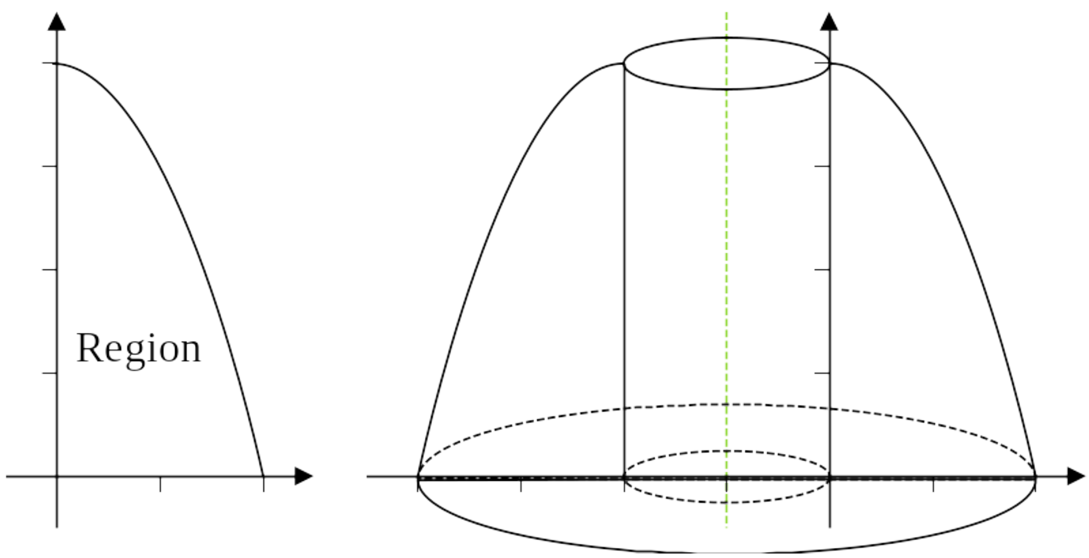
Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Summer 2021 (S62)
Quiz #4
Wednesday, 14 July.

Do all of the following questions. Show all your reasoning in each solution. Please note that part marks are available in questions worth more than 0.5 points, so incomplete or incorrect solutions may still earn something.

All the problems below refer to the region in the Cartesian plane below $y = 4 - x^2$ and above $y = 0$, where $0 \leq x \leq 2$.

1. Sketch the region and, separately, the solid obtained by revolving the region about the line $x = -1$. *[1]*

SOLUTION. Here are sketches of the region and the solid:



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2. Find the volume of the solid obtained by revolving the region about the line $x = -1$. *[2]*

SOLUTION. We will use the method of cylindrical shells to compute the volume of the solid. Since we revolved about the vertical line $x = -1$ and are using shells, the x-axis is perpendicular to the shells, so we will use x as our variable. Note if $0 \leq x \leq 2$, the shell at x (sketch it into the picture above yourself :-) has radius $r = x - (-1) = x + 1$ and height $h = y - 0 = y = 4 - x^2$, and thus has area $2\pi rh = 2\pi(x+1)\left(4 - x^2\right)$. It follows that the volume of the solid is given by:

$$
\begin{aligned}
V &= \int_0^2 2\pi r h \, dx = 2\pi \int_0^2 (x+1)\left(4-x^2\right) \, dx \\
&= 2\pi \int_0^2 \left(-x^3 - x^2 + 4x + 4\right) \, dx = 2\pi \left(-\frac{x^4}{4} - \frac{x^3}{3} + \frac{4x^2}{2} + 4x\right)\Bigg|_0^2 \\
&= 2\pi \left(-\frac{2^4}{4} - \frac{2^3}{3} + \frac{4 \cdot 2^2}{2} + 4 \cdot 2\right) - 2\pi \left(-\frac{0^4}{4} - \frac{0^3}{3} + \frac{4 \cdot 0^2}{2} + 4 \cdot 0\right) \\
&= 2\pi \left(-4 - \frac{8}{3} + 8 + 8\right) - 2\pi \cdot 0 = 2\pi \cdot \frac{28}{3} - 0 = \frac{56\pi}{3} \approx 58.6431 \quad \blacksquare
\end{aligned}
$$

3. Find the surface area of the solid (including the areas of its annular base and cylindrical hole) obtained by revolving the region about the line $x = -1$. *[2]*

Solution. The surface of the solid – check out the sketch again! – has three parts:

i. The base, which is a washer with outer radius $R = 2 - (-1) = 3$ and inner radius $r = 0 - (-1) = 1$, and hence has area $\pi\left(R^2 - r^2\right) = \pi\left(3^2 - 1^2\right) = 8\pi$.

ii. The hole in the middle, which is a cylinder of height $h = 4 - 0 = 4$ and radius $r = 0 - (-1) = 1$, and hence has area $2\pi r h = 2\pi \cdot 1 \cdot 4 = 8\pi$.

iii. The surface of revolution obtained by revolving the curve $y = 4 - x^2$, for $0 \leq x \leq 2$, about the line $x = -1$, whose area we compute below.

The increment of arc-length at x, $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$, when revolved in a circle of radius $r = x - (-1) = x + 1$, contributes an area of $2\pi r \, ds$ to the volume. In our case, $\frac{dy}{dx} = \frac{d}{dx}\left(4 - x^2\right) = -2x$, so the area of this surface is given by:

$$
SA = \int_0^2 2\pi r \, ds = \int_0^2 2\pi(x+1)\sqrt{1 + (-2x)^2} \, dx = 2\pi \int_0^2 (x+1)\sqrt{1 + 4x^2} \, dx
$$

We will split this integral up into two pieces, which we will tackle with different substitutions. For the first, we will use the substitution $u = 1 + 4x^2$, so $du = 8x \, dx$ and $x \, dx = \frac{1}{8} \, du$; for the second, we will use the trigonometric substitution $x = \frac{1}{2}\tan(\theta)$, so $dx = \frac{1}{2}\sec^2(\theta) \, d\theta$, and also $\tan(\theta) = 2x$ and $\sec(\theta) = \sqrt{1 + \tan^2(\theta)} = \sqrt{1 + 4x^2}$. We will keep the old limits in both cases and substitute back in terms of x after integrating before using them.

Here we go:

$$
\begin{aligned}
SA &= 2\pi \int_0^2 (x+1)\sqrt{1+4x^2}\,dx = 2\pi \left[\int x\sqrt{1+4x^2}\,dx + \int 1\sqrt{1+4x^2}\,dx \right]_0^2 \\
&= 2\pi \left[\int \sqrt{u}\,\frac{1}{8}\,du + \int \sqrt{1+4\cdot\frac{1}{4}\tan^2(\theta)}\;\frac{1}{2}\sec^2(\theta)\,d\theta \right]_0^2 \\
&= 2\pi \left[\frac{1}{8}\int u^{1/2}\,du + \frac{1}{2}\int \sqrt{\sec^2(\theta)}\;\sec^2(\theta)\,d\theta \right]_0^2 \\
&= 2\pi \left[\frac{1}{8}\cdot\frac{u^{3/2}}{3/2} + \frac{1}{2}\int \sec^3(\theta)\,d\theta \right]_0^2 \\
&= 2\pi \left[\frac{u^{3/2}}{12} + \frac{1}{2}\left(\frac{1}{2}\tan(\theta)\sec(\theta) + \frac{1}{2}\int \sec(\theta)\,d\theta \right) \right]_0^2 \\
&= 2\pi \left[\frac{u^{3/2}}{12} + \frac{1}{4}\left(\tan(\theta)\sec(\theta) + \ln\left(\tan(\theta)+\sec(\theta)\right)\right) \right]_0^2 \\
&= 2\pi \left[\frac{1}{12}\left(1+4x^2\right)^{3/2} + \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left(2x+\sqrt{1+4x^2}\right)\right) \right]_{)}^2 \\
&= \pi \left[\frac{1}{6}\left(\sqrt{1+4x^2}\right)^3 + x\sqrt{1+4x^2} + \ln\left(2x+\sqrt{1+4x^2}\right) \right]_0^2 \\
&= \pi \left[\frac{1}{6}\left(\sqrt{17}\right)^3 + 2\sqrt{17} + \ln\left(4+\sqrt{17}\right) \right] - \pi \left[\frac{1}{6}\left(\sqrt{1}\right)^3 + 0\sqrt{1} + \ln\left(0+\sqrt{1}\right) \right] \\
&= \pi \left[\frac{17\sqrt{7}}{6} + 2\sqrt{17} + \ln\left(4+\sqrt{17}\right) \right] - \pi \left[\frac{1}{6} + 0 + 0 \right] \\
&= \pi \left[\frac{17\sqrt{7}-1}{6} + 2\sqrt{17} + \ln\left(4+\sqrt{17}\right) \right]
\end{aligned}
$$

If you need a decimal, bring out your calculator and go!

Putting it all together, the surface area of the solid of revolution is

$$
\begin{aligned}
&8\pi + 8\pi + \pi \left[\frac{17\sqrt{7}-1}{6} + 2\sqrt{17} + \ln\left(4+\sqrt{17}\right) \right] \\
&= \pi \left[16 + \frac{17\sqrt{7}-1}{6} + 2\sqrt{17} + \ln\left(4+\sqrt{17}\right) \right].
\end{aligned}
$$

This can be simplified a bit, I suppose, but it's never going to be neat and this has gone on long enough ... Again, if you want a decimal, go to it! ■

[Total = 5]