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Power Series II - Tricks and Techniques

(A bit more of §11.8, plus §11.9, plus stuff not in the textbook.)

1° We can also write power series in terms of $(x-a)^n$ instead of x^n, where a is some constant.

<u>Def'n.</u> A power series centred at a $(a \in \mathbb{R})$ is one of the form $\sum_{n=0}^{\infty} a_n (x-a)^n$.

[Regular power series are power series centred at $a=0$.]

This can be used to write power series equal to some function when the regular power series does not converge at some point, or when the original function is undefined.

es $f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots$

$(R=1)$

This series converges only for $-1<x<1$, but $f(x)$ is defined for all $x \neq -1$. So if we want to have a series for $f(x)$ to be defined at $a=2$, we can simply expand around $a=2$:

$$f(x) = \frac{1}{1+x} = \frac{1}{3+(x-2)} = \frac{1}{3+(x-2)} \cdot \frac{\frac{1}{3}}{\frac{1}{3}} = \frac{\frac{1}{3}}{1+\underbrace{\tfrac{1}{3}(\cancel{x}x-2)}}$$

$$= \frac{\overset{\text{first term}}{\frac{1}{3}}}{1-\underbrace{\left(-\frac{1}{3}(x-2)\right)}_{r}} = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3^{n+1}}$$

This has radius of convergence $R=3$: it's a geometric series, so it converges for $\left|\frac{(-1)^n(x-2)^n}{3^{n+1}}\right| < 1$

$\iff |x-2|^n < 3^{n+1}$

$\iff |x-2| < 3$ & diverges otherwise

So $R=3$ & the "interval of cvgce." is $(2-3, 2+3) = (-1, 5)$.

Note that the original function has a vertical asymptote at $x=-1$. In general, if you expand around a and have a VA at b, then $R \leq |b-a|$. (A power series expansion can only work up to the nearest vertical asymptote at best.) (3)

eg ⇒ $f(x) = \ln(x)$ cannot be expanded as a power series about $a=0$ since $\ln(x)$ is undefined at 0 (& has a VA there)

... but $\ln(x) = \int \frac{1}{x}\,dx = \int \frac{1}{1-(-(x-1))}\,dx$

$$= \int \left(1-(x-1)+(x-1)^2-(x-1)^3+\cdots\right)dx \qquad \begin{array}{l} u=x-1 \\ du=dx \end{array}$$

$$= \int \left(1-u+u^2-u^3+\cdots\right)du = C+u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\cdots$$

$$= \cancel{C}+(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots$$

To get C, plug in $x=1$ on both sides so $\ln(1)=0=C+0$ $\Rightarrow C=0$

This last illustrates another common trick:

2° You can safely differentiate or integrate a power series term-by-terms within the radius of convergence. The new series will have the same radius of cvgce., but may be different at the endpoints $a \pm R$, radius of cvgce

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$R = 1$ & centre $a = 0$.

Then

$$f'(x) = \frac{d}{dx}(1+x^2)^{-1} = -(1+x^2)^{-2} \cdot \frac{d}{dx}(1+x^2) = \frac{-1}{(1+x^2)^2} \cdot 2x$$

$$= \frac{-2x}{(1+x^2)^2}$$

can be expressed in terms of a power series by differentiating the power series for $f(x)$:

$$\frac{d}{dx}\left(1 - x^2 + x^4 - x^6 + \cdots\right) = 0 - 2x + 4x^3 - 6x^5 + 8x^7 - 10x^9 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1}(2n+2)\, x^{2n+1}$$

3° Within the radius of convergence of two power series (centred at the same point) we can add, subtract, & multiply by constants to get a new power series from the given ones. The new radius of convergence will be the smaller of the two you started with.

$\Rightarrow$ $f(x) = \dfrac{1}{1-x} + \dfrac{2}{1+x^2}$ (both have $R=1$)

$$= \left(1 - x + x^2 - x^3 + x^4 - \cdots\right) + \left(2 - 2x^2 + 2x^4 - 2x^6 + \cdots\right)$$

$$= 3 - x - x^2 - x^3 + 3x^4 - x^5 - x^6 - x^7 + 3x^8 - \cdots$$

Note: Perversely, this kind of thing is most useful when dealing with discrete mathematics, where the coefficients of infinite series are used to keep track of various ways of counting arrangements of finitely many objects.

This can also be extended to multiplication of series.

eg $\cancel{\frac{d}{(1+x)^2}}$

$$\frac{1}{(1-x)^2} = \left(1+x+x^2+x^3+\cdots\right)^2$$

$$= \left(1+x+x^2+x^3+\cdots\right)\left(1+x+x^2+x^3+\cdots\right)$$

$$\begin{aligned}
= \; & 1+x+x^2+x^3+x^4+\cdots \\
& \quad +x+x^2+x^3+x^4+\cdots \\
& \qquad +x^2+x^3+x^4+\cdots \\
& \qquad\quad +x^3+x^4+\cdots \\
& \qquad\qquad \vdots
\end{aligned}$$

$$= 1+2x+3x^2+4x^3+5x^4+\cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

Alternatively,

$$\frac{d}{dx}\frac{1}{1-x} = \frac{-1}{(1-x)^2}\cdot\frac{d}{dx}(1-x) = \frac{(-1)(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$= \frac{d}{dx}\left(1+x+x^2+\cdots\right) = 0+1+2x+3x^2+\cdots$$

Next: Taylor series.