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Sequences and Their Limits

(§11.1 in the text)

A sequence is a list of numbers, indexed by the integers from some point on (most often 0 or 1).

eg

index	0	1	2	3	4	$\cdots$	n	$\cdots$	$\to\infty$
sequence	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$	$\to 0$

Notation: We usually describe the sequence by something like $\{a_n \mid n \geq k\}$ or $\{a_n\}_k^\infty$ or $\{a_n\}$ ← especially if you don't really care where it starts.

eg In the above example $k=0$ and $a_n = \frac{1}{2^n}$, so you might write $\{\frac{1}{2^n} \mid n \geq 0\}$ or $\{\frac{1}{2^n}\}$.

We'll usually be most interested in where the sequence is going, ie in its limit as $n\to\infty$. This will be defined in a way similar to the way we define $\lim_{x\to\infty} f(x)$. (The horror of epsilonics strikes again.)

<u>Def'n</u>: The <u>limit</u> of a sequence $\{a_n\}$ is L, written as $\lim_{n\to\infty} a_n = L$ or as $a_n \to L$; means:

For every $\varepsilon > 0$, there is an N_ε, such that for all $n \geq N_\varepsilon$, we have that $|a_n - L| < \varepsilon$.

<u>eg</u> $\{\frac{1}{2^n}\}$ - we claim this has limit 0

<u>ie</u> For every $\varepsilon > 0$, there is an N_ε, s.t. for all $n \geq N_\varepsilon$, $|\frac{1}{2^n} - 0| < \varepsilon$.

Given an $\varepsilon > 0$, we find a suitable N_ε by reverse-engineering it from $|\frac{1}{2^n} - 0| < \varepsilon$.

$$\left|\frac{1}{2^n} - 0\right| < \varepsilon \iff \frac{1}{2^n} < \varepsilon$$
$$\iff 1 < \varepsilon \cdot 2^n$$
$$\iff \frac{1}{\varepsilon} < 2^n$$
$$\iff \log_2\left(\frac{1}{\varepsilon}\right) < n.$$

Every step is reversible here.

So if we let $N_\varepsilon = \log_2\left(\frac{1}{\varepsilon}\right)$, then if $n \geq N_\varepsilon$, then $\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} < \varepsilon$.

Thus, according to the "$\varepsilon - N_\varepsilon$" definition of the limit of a sequence, $\lim_{n \to \infty} \frac{1}{2^n} = 0$.

In practice, using this definition directly is cumbersome, so we rely on general properties of limits to actually compute them.

Some of the basic rules for limits of sequences:

1. $\lim_{n\to\infty} c a_n = c\left(\lim_{n\to\infty} a_n\right)$ (Assuming $\lim_{n\to\infty} a_n$ exists.)

2. $\lim_{n\to\infty} (a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right)$ (Assuming the limits exist.)

3. $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$ (—— " ——)

4. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \left(\lim_{n\to\infty} a_n\right) \Big/ \left(\lim_{n\to\infty} b_n\right)$ (—— " —— and we don't divide by 0 - with the exception that we don't care if finitely many b_n are $= 0$)

(& similarly for other properties like limits of functions)

\+ 5. If $a_n = f(n)$, where $f(x)$ is a continuous function, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x),$$ assuming $\lim_{x\to\infty} f(x)$ exists.

Potential glitch: It's possible for $\lim_{n\to\infty} f(n)$ to exist, even if $\lim_{x\to\infty} f(x)$ does not.

eg $f(x) = \sin(\pi x)$

Then $a_n = f(n)$ (for $n \geq 0$)

$= \sin(n\pi) = 0$

Obviously, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \lim_{n\to\infty} 0 = 0$

but $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin(\pi x)$ doesn't exist

(because $\sin(\pi x)$ oscillates between -1 & 1 forever).

The use of the continuous counterpart (if there is one) let's us, in particular, employ l'Hôpital's Rule.

eg $a_n = \dfrac{n^2}{2^n}$ ($n \geq 0$)

The continuous function corresponding to this would be $f(x) = \dfrac{x^2}{2^x}$. So

$$\lim_{n\to\infty} \frac{n^2}{2^n} = \lim_{x\to\infty} \frac{x^2 \to \infty}{2^x \to \infty} = \lim_{x\to\infty} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} 2^x} = \lim_{x\to\infty} \frac{2x \to \infty}{\ln(2)\cdot 2^x \to \infty}$$

provided exists

So we can apply l'Hôpital's Rule.

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}(\ln(2)\cdot 2^x)} = \lim_{x\to\infty} \frac{2}{\ln(2)\cdot\ln(2)\cdot 2^x} \begin{matrix}\to 2\\ \to\infty\end{matrix}$$

$$= 0$$

So $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$.

We can't use this (rule 5) if we can't find a continuous counterpart to our sequence or when we can't handle that continuous counterpart.

$\underline{\text{es}}$ $\quad a_n = \frac{\lceil 2^n+3^n \rceil}{2.5^n}$ $\leftarrow$ smallest ~~largest~~ integer $\geq \frac{2^n+3^n}{2.5^n}$ has not got a nice continuous counterpart

$a_n = \frac{1}{n!}$ $\quad$ has a continuous counterpart, namely the Gamma function

$n! = n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1$ $\quad$ $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$ $\quad$ but at this stage we're not up to handling it or it's derivative

$n! = \Gamma(n+1)$

Some hopefully not unfamiliar tools from regular limits that we can still use if the continuous counterpart trick (rule 5) fails to be useful:

6 The Squeeze Theorem

If $a_n \le b_n \le c_n$ for all n past some point,

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ (and both limits exist),

then $\lim_{n\to\infty} b_n$ exists and equals L too.

eg $a_n = \frac{2^n}{n!}$ $(n \ge 0)$ [$0! = 1$, by def'n]

$$0 \le \frac{2^n}{n!} \le \frac{2\cdot 2\cdot 2\cdot \ldots \cdot 2\cdot 2}{n(n-1)(n-2)\cdot\ldots\cdot 2\cdot 1} \le \frac{2\cdot 2}{n\cdot 1} = \frac{4}{n}$$

≤ 1

Since $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{4}{n}$ $\begin{smallmatrix}\to 4\\ \to\infty\end{smallmatrix}$, we can conclude

that $\lim_{n\to\infty} \frac{2^n}{n!}$ is also $= 0$.

(7) Monotone Convergence Theorem

Suppose $\{a_n\}$ is a non-decreasing [respectively, non-increasing] sequence that has an upper bound [respectively, lower bound]. Then $\lim_{n\to\infty} a_n$ exists. [The limit will be the ~~greatest~~ least upper bound of the sequence. [respectively, the greatest lower bound]]

eg $a_n = \frac{1}{2^n}$ $(n \geq 0)$ we have

$$1 > \frac{1}{2} > \frac{1}{4} > \dots > \frac{1}{2^n} > \frac{1}{2^{n+1}} > \dots > 0$$

the obvious lower bound.

So the Monotone Convergence Theorem tells us $\lim_{n\to\infty} \frac{1}{2^n}$ exists (but showing that limit is actually 0 requires doing something else.)