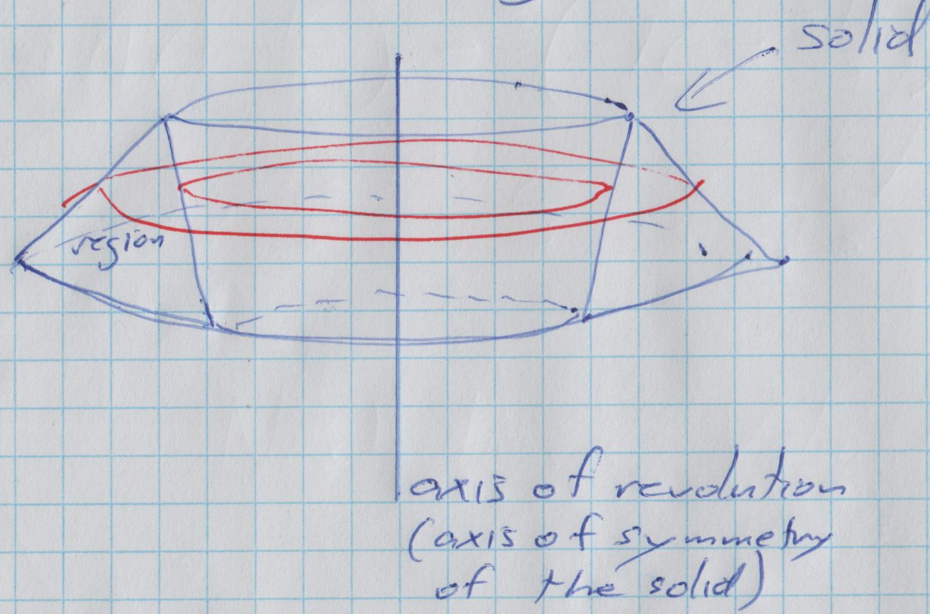
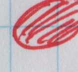
Volumes II - volumes of solids of revolution

(a bit more than is done in §9.3 of the textbook)

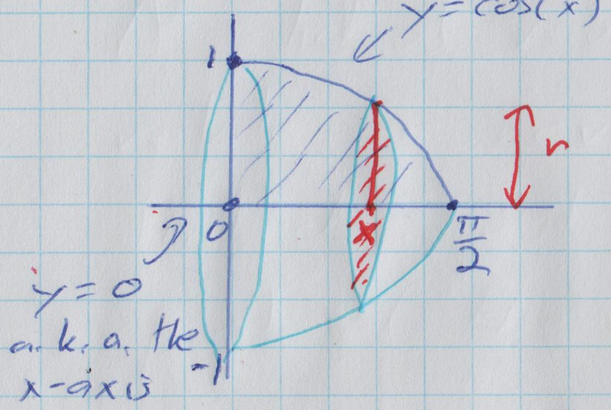
A solid of revolution is what you get when you rotate or revolve a 2-D region about a line that does not pass inside the region (it may be part of the region's border).



The textbook only considers situations where the axis of revolution is the x-axis or the y-axis. We'll do this where it could be any vertical or horizontal line.

Cross-sections perpendicular to the axis of revolution are circular: disk  or annulus .

es The region is the one below $y=\cos(x)$ and above $y=0$ for $0 \le x \le \frac{\pi}{2}$. We'll revolve this about the x-axis



"Areas of cross-sections sweep out volume."

The cross-section at x is a disk (a circle plus its interior), so

$A(x) = \pi r^2 = \pi \cos^2(x)$

because $r = \cos(x) - 0 = \cos(x)$.

$$V = \int_0^{\pi/2} \pi r^2\, dx = \pi \int_0^{\pi/2} \cos^2(x)\, dx = \pi \int_0^{\pi/2} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\right) dx$$

$u = 2x$
$du = 2dx$
$dx = \frac{1}{2} du$

x	u
0	0
$\pi/2$	π

We use x as the variable here because the x-axis is the one perpendicular to the cross-sections.

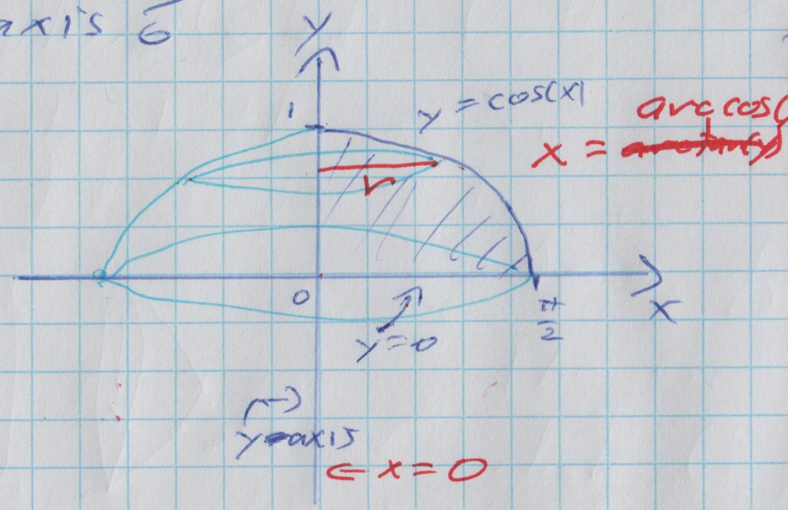
$$= \frac{\pi}{2} \int_0^{\pi} (1 + \cos(u))\, \tfrac{1}{2}\, du$$

$$= \frac{\pi}{4} \left(u + \sin(u)\right) \Big|_0^{\pi}$$

$$= \frac{\pi}{4}(\pi + \underbrace{\sin(\pi)}_{0}) - \cancel{\frac{\pi}{4}(0 + \underbrace{\sin(0)}_{0})}$$

$$= \frac{\pi^2}{4}$$

How about revolving the same region about the y-axis?



The cross-sections are perpendicular to the y-axis, so we ought to use y as the variable, so we have to describe the region in terms of y.

The region is given by

$$0 \le x \le \arccos(y)$$

with the radius of the cross-section at y given by

$$r = x - 0 = \arccos(y) - 0 = \arccos(y).$$

for $0 \le y \le 1$.

Then the volume integral using the circular cross-sections is given by

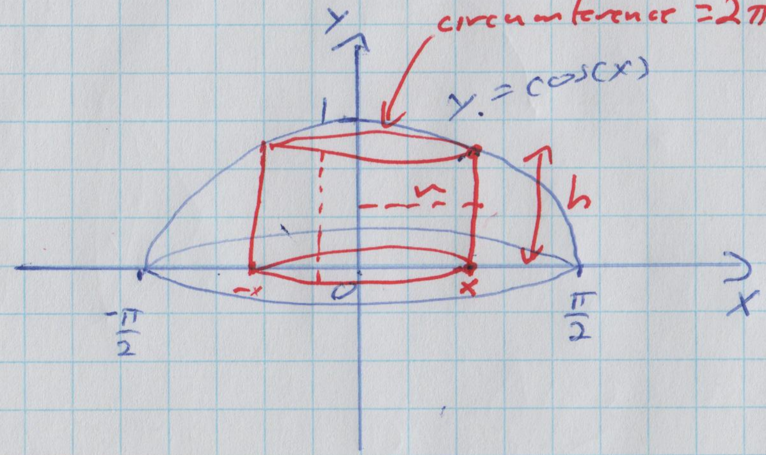
$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi r^2\,dy$$

$$= \pi \int_0^1 \arccos^2(y)\,dy$$

This *could* be done with integration by parts, etc..

... but it's a *pain*.

Can we find a better way? This boils down to finding an alternative to using circular cross-sections. There is such an alternative.

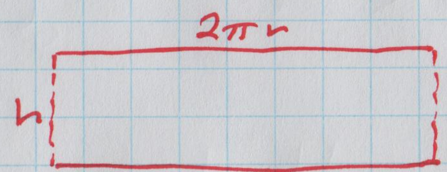


A cross-section at x perpendicular to the x-axis (so we can use x as a variable, looks like a vertical line segment in the original region.

Revolved about the axis of revolution this line segment becomes a cylinder, without the caps on top bottom & not counting the inside of the cylinder.

The cylinder has radius $r = x - 0 = x$ and height $h = y - 0 = \cos(x) - 0 = \cos(x)$,

so it has area (unroll!)



$$A(x) = 2\pi r h = 2\pi x \cos(x).$$

Thus the volume is given by

$$V = \int_0^{\pi/2} A(x)\,dx = \cancel{\int_0^{\pi/2} \pi r^2\,dx}\ \int_0^{\pi/2} 2\pi r h\,dx$$

Use parts:
$u = x \quad v = \cos(x)$
$u' = 1 \quad v = \sin(x)$

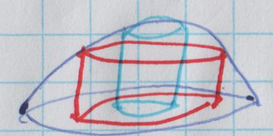
$$= \int_0^{\pi/2} 2\pi x \cos(x)\,dx = 2\pi \int_0^{\pi/2} x\cos(x)\,dx$$

$$= 2\pi \left(x\sin(x) \Big|_0^{\pi/2} - \int_0^{\pi/2} 1\cdot \sin(x)\,dx \right)$$

$$= 2\pi \left[\underbrace{\left(\frac{\pi}{2}\cdot \sin\left(\frac{\pi}{2}\right) - 0\cdot \sin(0) \right)}_{0} + \underbrace{(+\cos(x))\Big|_0^{\pi/2}}_{1} \right]$$

$$= 2\pi \left[\frac{\pi}{2}\cdot 1 - 0 + \left(\cos\left(\frac{\pi}{2}\right) - \cos(0) \right) \right]$$

$$= 2\pi \left[\frac{\pi}{2} - 1 \right] = \pi^2 - 2\pi .$$



So we have two alternative methods – cross-section perpendicular to the axis of revolution (disks or washers (annuli)) or parallel to the axis of revolution (cylindrical shells). More next time!