# "Improper" Integrals, or integrating around asymptotes of the integrand

(§9.7 in the text, mixed up with applications I'm ignoring here...)

Our definition of definite integrals really only works if the function has no vertical asymptotes in the relevant and the relevant interval is finite.

Q: How do we deal with an integral ("definite"?)

$$\int_0^{\infty} 2e^{-2x}\,dx = \lim_{a\to\infty} \int_0^a 2e^{-2x}\,dx$$

A: Use limits! (Review...)

$u = -2x \quad du = -2dx$

$\Rightarrow (-1)du = (+2)dx$

| x | u |
|---|---|
| 0 | 0 |
| a | -2a |

$$= \lim_{a\to\infty} \int_0^{-2a} e^u (-1)\,du = \lim_{a\to\infty} \int_{-2a}^{0} e^u\,du = \lim_{a\to\infty} e^u \Big|_{-2a}^{0}$$

$$= \lim_{a\to\infty} \left(e^0 - e^{-2a}\right) = \lim_{a\to\infty} \Big(1 - \underbrace{\boxed{\frac{1}{e^{2a}}}}_{\to 0}\Big) = 1 - 0 = 1$$

(This was the positive part of the exponential ~~distribution~~ density function with $\lambda = 2$ being integrated.

(Used in stats & prob.)

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$ )

<u>Terminology:</u> An improper integral that works out a real number <u>converges</u>; one that does not <u>diverges</u>.

(Borrowed from that used with infinite series.)

<u>Example:</u> $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty} \ln(x)\Big|_1^a$

$\displaystyle = \lim_{a\to\infty} \big(\ln(a) - \underbrace{\ln(1)}_{=0}\big) = \lim_{a\to\infty} \ln(a) = \infty$

so this integral diverges.

<u>Note:</u> We can't toss $\infty$ like a real number, so

$\displaystyle\int_1^\infty \frac{1}{x}\,dx = \ln(x)\Big|_1^\infty = \ln(\infty) - \ln(1) = \infty - 0 = \infty$ is <u>not</u> ok.

Example: $\displaystyle\int_{-\infty}^{\infty} \frac{4x}{1+x^2}\,dx$ (3)

If you have an asymptote at both ends, split the integral in the middle.

$$= \int_{-\infty}^{0} \frac{4x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{4x}{1+x^2}\,dx \qquad \begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \end{aligned}$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{4x}{1+x^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{4x}{1+x^2}\,dx$$

| $x$ | $u$ |
|---|---|
| $a$ | $1+a^2$ |
| $0$ | $1$ |

| $x$ | $u$ |
|---|---|
| $0$ | $1$ |
| $b$ | $1+b^2$ |

$$= \lim_{a\to-\infty} \int_{1+a^2}^{1} \frac{2}{u}\,du + \lim_{b\to\infty} \int_{1}^{1+b^2} \frac{2}{u}\,du$$

$$= \lim_{a\to-\infty} 2\ln(u)\Big|_{1+a^2}^{1} + \lim_{b\to\infty} 2\ln(u)\Big|_{1}^{1+b^2}$$

$$= \lim_{a\to-\infty} \left(2\overset{0}{\ln(1)} - 2\ln(1+a^2)\right) + \lim_{b\to\infty} \left(2\ln(1+b^2) - 2\overset{0}{\ln(1)}\right)$$

$$= \lim_{a\to-\infty} \underbrace{-2\ln\underbrace{(\overbrace{1+a^2}^{\to\infty})}_{\to\infty}}_{\to-\infty} + \lim_{b\to\infty} \underbrace{2\ln\underbrace{(\underbrace{1+b^2}_{\to\infty})}_{\to\infty}}_{\to\infty}$$

$$= -\infty + \infty$$

It's tempting to proceed:

$$-\infty + \infty = 0$$

This a <u>no-no</u>...

If you get an infinity out of either part (or both parts) in a split improper integral calculation the original integral works out to: "diverges".

It's important to do the split because otherwise you get $\int_{-\infty}^{\infty} \frac{4x}{1+x^2}\,dx = \lim_{a\to\infty} \int_{-a}^{a} \frac{4x}{1+x^2}\,dx = \lim_{a\to\infty} \int_{1+a^2}^{1+a^2} \frac{2}{u}\,du$

$$= \lim_{a\to\infty} 0 = \cancel{0}$$

integral over a single point.

If we have a Vertical Asymptote ("VA") at an endpoint of the interval, we use limits as well.

Example: $\int_0^1 x^{-1/3}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$ as $x \to 0^+$, $\frac{1}{\sqrt[3]{x}} \to +\infty$

$$= \lim_{a\to 0^+} \int_a^1 x^{-1/3}\,dx = \lim_{a\to 0^+} \left.\frac{x^{2/3}}{2/3}\right|_a^1$$

$$= \lim_{a\to 0^+} \left.\frac{3}{2}x^{2/3}\right|_a^1 = \lim_{a\to 0^+}\left(\frac{3}{2}\cdot 1^{2/3} - \frac{3}{2}\cdot a^{2/3}\right)$$

$$= \lim_{a\to 0^+}\left(\frac{3}{2} - \frac{3}{2}a^{2/3}\right) = \frac{3}{2} - \frac{3}{2}\cdot 0 = \frac{3}{2}$$

since $2/3 > 0$
$a^{2/3} \to 0^+$
as $a \to 0^+$

Similarly, $\int_0^1 x^3\,dx = \lim_{a\to 0^+} \int_a^1 x^3\,dx = \lim_{a\to 0^+} \left.\frac{x^4}{4}\right|_a^1$

not an improper integral.

$$= \lim_{a\to 0^+}\left(\frac{1^4}{4} - \underbrace{\frac{a^4}{4}}_{\to\, 0}\right) = \frac{1}{4}$$

Moral: if you mistake a "proper" integral for an "improper" integral you'll only do extra work.

A horrendously terrible & terribly horrendous example with intimidating integrand and both kinds of asymptotes:

$$\int_0^{\infty} -2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)e^{-\left(x+\frac{1}{x}\right)^2}\,dx$$

Problems at both ends, so we split the integral.

$$= \int_0^{1} -2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)e^{-\left(x+\frac{1}{x}\right)^2}\,dx + \int_1^{\infty} -2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)e^{-\left(x+\frac{1}{x}\right)^2}\,dx$$

In both integral, we will eventually use the substitution $u = -\left(x+\frac{1}{x}\right)^2$

$$du = -2\left(x+\frac{1}{x}\right)\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$
$$= -2\left(x+\frac{1}{x}\right)\cdot\left(1+\frac{-1}{x^2}\right)$$
$$= -2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)$$

| $x$ | $u$ |
|---|---|
| $a$ | $-\left(a+\frac{1}{a}\right)^2$ |
| $1$ | $-4$ |

| $x$ | $u$ |
|---|---|
| $1$ | $-4$ |
| $b$ | $-\left(b+\frac{1}{b}\right)^2$ |

$$= \lim_{a\to 0^+}\int_a^{1} -2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)e^{-\left(x+\frac{1}{x}\right)^2}\,dx + \lim_{b\to\infty}\int_1^{b} -2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)e^{-\left(x+\frac{1}{x}\right)^2}\,dx$$

$$= \lim_{a\to 0^+} \int_{-(a+\frac{1}{a})^2}^{-4} e^u\,du + \lim_{b\to\infty} \int_{-4}^{-(b+\frac{1}{b})^2} e^u\,du$$

(antiderivatives are both $e^u$)

$$= \lim_{a\to 0^+}\left[e^{-4} - \underbrace{e^{-(a+\frac{1}{a})^2}}_{\to 0^+}\right] + \lim_{b\to\infty}\left[\underbrace{e^{-(b+\frac{1}{b})^2}}_{\to 0^+} - e^{-4}\right]$$

as $a \to 0^+$,

$a + \frac{1}{a} \to +\infty$,

so $-(a+\frac{1}{a})^2 \to -\infty$

so

as $b \to \infty$,

$b + \frac{1}{b} \to +\infty$,

so $-(b+\frac{1}{b})^2 \to -\infty$

so

$$= (e^{-4} - 0) + (0 - e^{-4}) = e^{-4} - e^{-4} = 0$$

So this integral actually works out to 0.

[Not many convergent improper integrals do, but it does happen.]