Trigonometric Substitutions II

- handling general quadratics

From last time: $\int \sqrt{x^2+3x+17}\,dx$

We need to do some algebraic preparation and a preliminary substitution before we get into trig substitution.

First, we "complete the square" on $x^2+3x+17$

$$x^2+3x+17 = \left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + 17$$
$$= \left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + \frac{68}{4}$$
$$= \left(x+\frac{3}{2}\right)^2 + \frac{59}{4}$$

$$x^2+bx+c$$
$$= \left(x+\frac{b}{2}\right)^2 - \frac{b^2}{4} + c$$
$$= x + \cancel{2}\frac{b}{\cancel{2}}x + \cancel{\frac{b^2}{4}} - \cancel{\frac{b^2}{4}} + c$$

Second, we substitute $u = x + \frac{3}{2}$, so $du = dx$

$$\int \sqrt{x^2+3x+17}\,dx = \int \sqrt{\left(x+\frac{3}{2}\right)^2 + \frac{59}{4}}\,dx$$

$$= \int \sqrt{u^2 + \frac{59}{4}}\,dx$$

$\sqrt{\frac{59}{4}} = \frac{\sqrt{59}}{2}$

$u = \sqrt{\frac{59}{4}}\tan(\theta)$

$= \frac{\sqrt{59}}{2}\tan(\theta)$

Third, we do the trig substitution (at last!) ...

$$= \int \sqrt{\frac{59}{4}\tan^2(\theta) + \frac{59}{4}} \cdot \frac{\sqrt{59}}{2}\sec^2(\theta)\,d\theta$$

$du = \frac{\sqrt{59}}{2}\sec^2(\theta)\,d\theta$

$$= \int \frac{\sqrt{59}}{2}\sqrt{\underbrace{\tan^2(\theta)+1}_{\sec^2(\theta)}} \cdot \frac{\sqrt{59}}{2}\sec^2(\theta)\,d\theta$$

$$= \frac{59}{4}\int \sec(\theta)\cdot\sec^2(\theta)\,d\theta$$

$$= \frac{59}{4}\int \sec^3(\theta)\,d\theta$$

& now proceed as before ...

[as an exercise, finish this]

An example without a square root...

$$\int \frac{2x+6}{4x^2+8x+24}\,dx = \int \frac{\cancel{2}(x+3)}{\cancel{2}(2x^2+4x+6)}\,dx$$

$u = 2x^2+4x+6$

$du = 4x+4$

$= 4(x+1)\,dx$

$\frac{1}{4}du = (x+1)\,dx$

$$= \int \frac{x+1+2}{2x^2+4x+6}\,dx = \int \frac{x+1}{2x^2+4x+6}\,dx + \int \frac{2}{2x^2+4x+6}\,dx$$

$$= \int \frac{1}{u}\cdot\frac{1}{4}\,du + \int \frac{\cancel{2}\,1}{\cancel{2}(x^2+2x+3)}\,dx$$

x^2+2x+3

$= \left(x+\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + 3$

$= (x+1)^2 - 2$

$$= \frac{1}{4}\ln(u) + \int \frac{1}{(x+1)^2-2}\,dx$$

$w = x+1 \quad dw = dx$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \int \frac{1}{w^2-2}\,dw$$

$w = \sqrt{2}\sec(\theta)$

$dw = \sqrt{2}\sec(\theta)\tan(\theta)\,d\theta$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \int \frac{1}{2\sec^2(\theta)-2}\cdot\sqrt{2}\sec(\theta)\tan(\theta)\,d\theta$$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{\sqrt{2}}{2}\int \frac{1}{\sec^2(\theta)-1}\sec(\theta)\tan(\theta)\,d\theta$$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\int \frac{1}{\tan^{\cancel{2}}(\theta)}\sec(\theta)\cancel{\tan(\theta)}\,d\theta$$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\int \frac{\sec(\theta)}{\tan(\theta)}\,d\theta$$

When all else fails in a trig integral, rewrite in terms of sin & cos & see what you can do

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\int \frac{\frac{1}{\cos(\theta)}}{\frac{\sin(\theta)}{\cos(\theta)}}\,d\theta$$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\int \frac{1}{\cancel{\cos(\theta)}}\cdot\frac{\cancel{\cos(\theta)}}{\sin(\theta)}\,d\theta$$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\int \frac{1}{\sin(\theta)}\,d\theta$$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\int \csc(\theta)\,d\theta$$

& now what?
You can integrate this with a trick similar to integrating sec(θ).

The smarter thing to do (if you can)
is to look it up (or hand it off to software).

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\ln(\csc(\theta)-\cot(\theta)) + C$$

$\sec(\theta) = \frac{1}{\cos(\theta)}$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\ln\left(\frac{1}{\sin(\theta)} - \frac{\cos(\theta)}{\sin(\theta)}\right) + C$$

$\Rightarrow \cos(\theta) = \frac{1}{\sec(\theta)}$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\ln\left(\frac{\sec(\theta)}{\tan(\theta)} - \frac{1}{\tan(\theta)}\right) + C$$

$\frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\tan(\theta)}$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\ln\left(\frac{\sec(\theta)-1}{\tan(\theta)}\right) + C$$

$\sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{1-\frac{1}{\sec^2(\theta)}} = \frac{\sqrt{\sec^2(\theta)-1}}{\sec(\theta)} = \frac{\sqrt{\tan^2(\theta)}}{\sec(\theta)} = \frac{\tan(\theta)}{\sec(\theta)}$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\ln\left(\frac{\frac{w}{\sqrt{2}}}{\sqrt{\underbrace{\sec^2(\theta)}_{\frac{w^2}{2}}-1}}\right) + C$$

$1 = \left(\frac{\sqrt{2}}{\sqrt{2}}\right)^2 = \frac{2}{2}$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\ln\left(\frac{\cancel{\frac{1}{\sqrt{2}}}\cdot w}{\cancel{\frac{1}{\sqrt{2}}}\sqrt{w^2-2}}\right) + C$$

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\ln\left(\frac{x+1}{\sqrt{(x+1)^2-2}}\right) + C$$

6

$$= \frac{1}{4}\ln(2x^2+4x+6) + \frac{1}{\sqrt{2}}\ln\left(\frac{x+1}{\sqrt{x^2+2x-1}}\right) + C$$

$(x^2+2x-1)^{1/2}$

$$= \frac{1}{4}\ln(2(x^2+2x+3)) + \frac{1}{\sqrt{2}}\ln(x+1) - \frac{1}{\sqrt{2}}\ln\left(\sqrt{x^2+2x-1}\right) + C$$

$$= \frac{1}{4}\ln(2) + \frac{1}{4}\ln(x^2+2x+3) + \frac{1}{\sqrt{2}}\ln(x+1) - \frac{1}{2\sqrt{2}}\ln(x^2+2x-1) + C$$

$$= \frac{1}{4}\ln(x^2+2x+3) + \frac{1}{\sqrt{2}}\ln(x+1) - \frac{1}{2\sqrt{2}}\ln(x^2+2x-1) + D$$

I'd probably stop around unless I had a good reason to go on...

Moral: apparently simple things can take a while when integrating.