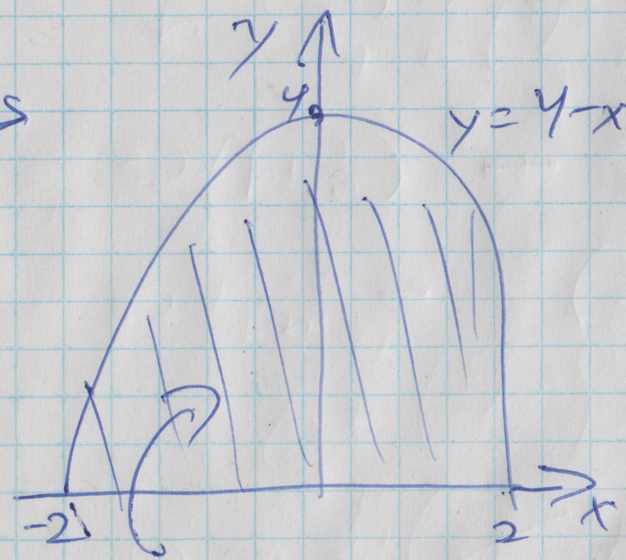
The Fundamental Theorem of Calculus

... preceded by a bit of weirdness: negative areas.

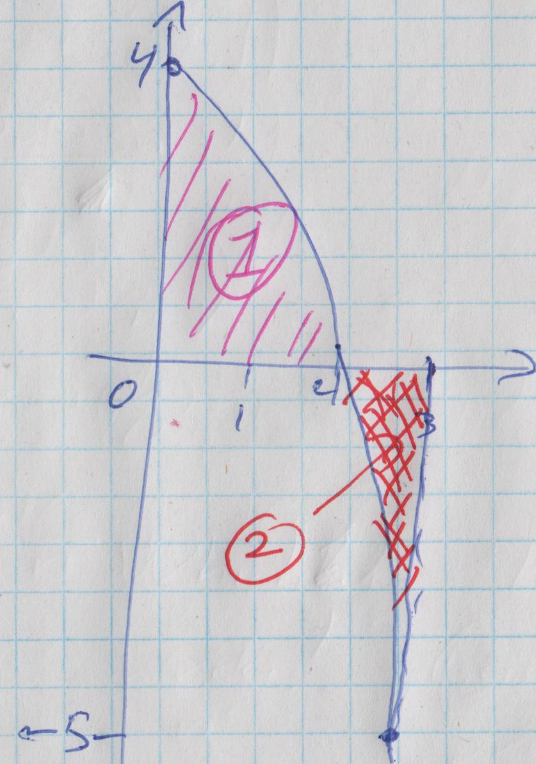
Notation: We use $\int_a^b f(x)\,dx$ "the definite integral of $f(x)$ from a to b"

to denote the area between the graph of $y=f(x)$ and the x-axis for $a \le x \le b$.

eg



$$\text{Area} = \int_{-2}^{2} (4-x^2)\,dx$$



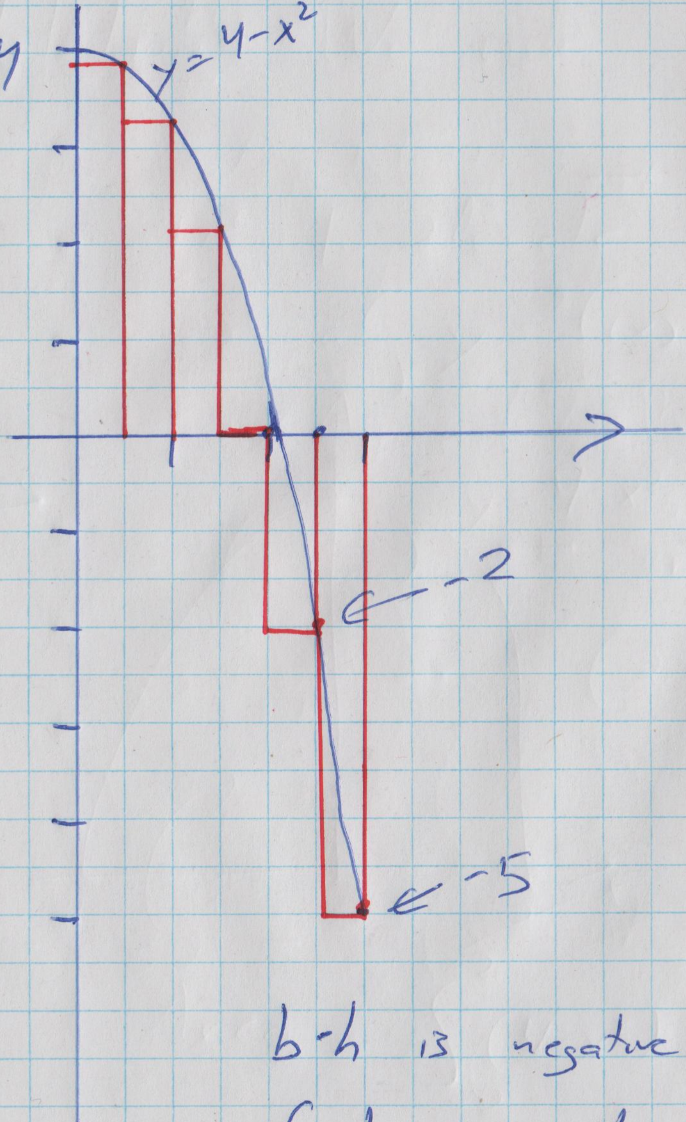
$$\int_0^3 (4-x^2)\,dx$$

$=$ Area 1

$-$ Area 2

Area below the x-axis is negative.

Why? It's a consequence of using rectangles to approximate areas and using the function to find the heights of the rectangles:



$b \cdot h$ is negative if h is negative... $(b > 0)$

This is a problem if you want the <u>area</u> <u>in a normal sense</u>: you have to break up the region & handle the parts below the x-axis separately.

The Fundamental Theorem of Calculus

I. Suppose that $F(x)$ is a function on $[a,b]$ which is differentiable at every point $x \in (a,b)$ and such that $f(x) = F'(x)$ is bounded [no vertical asymptotes] on (a,b). Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

[This is the version we'll use most of the time]

assuming the definite integral makes sense ("integrable")
[which is guaranteed if, for example, $f(x)$ is continuous]

II. Suppose that $f(x)$ is defined and integrable on $[a,b]$. Then if we define $F(x)$ by $F(x) = \int_a^x f(t)\,dt$,

we have that $F(x)$ is differentiable on (a,b) and

$$F'(x) = f(x).$$

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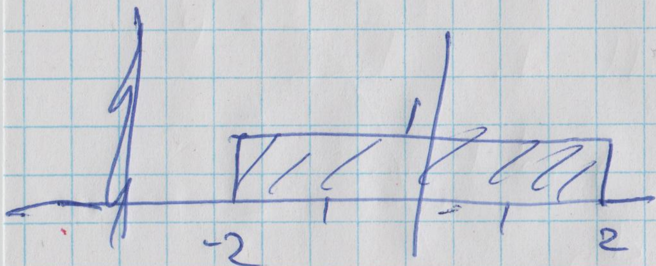
$\int_{-2}^{2} 1\,dx$: 1 is the derivative of $f(x) = x$

so

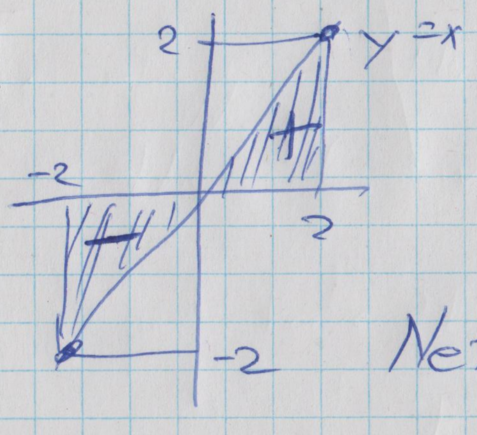
$$\int_{-2}^{2} 1\,d(x) = x\Big|_{-2}^{2} = 2-(-2) = 4$$

"x evaluated between -2 and 2"



$\int_{-2}^{2} x\,dx$: x is the derivative of $\frac{x^2}{2}$; $\frac{d}{dx}\left(\frac{x^2}{2}\right) = \cancel{\frac{1}{2}}\cdot\cancel{2}x = x$

so



$$\int_{-2}^{2} x\,dx = \frac{x^2}{2}\Big|_{-2}^{2} = \frac{2^2}{2} - \frac{(-2)^2}{2} = \frac{4}{2} - \frac{4}{2} = 0$$

Net area $= \frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}\cdot 2\cdot 2$
$= 0$

Our problem using the Fundamental Theorem (I) (5)
is that we need to find <u>anti-derivatives</u>.
We develop rules for finding them, plus a library of basic ones

1° <u>Power Rule:</u> the antiderivative of x^n is $\frac{x^{n+1}}{n+1}$ (since $\frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{\cancel{(n+1)}x^n}{\cancel{n+1}} = x^n$)
except when $n=-1$. When $n=-1$, the antiderivative is $\ln(x)$, because $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

2. <u>Library of basic trig functions:</u>

"indefinite integral" which works out to a generic antiderivative

$\frac{d}{dx}\sin(x) = \cos(x)$ so $\int \cos(x)\,dx = \sin(x) + C$

"constant of integration"

Assumed to be 0 for ~~def~~ computing definite integrals, since it cancels out anyway.

Since $\frac{d}{dx} C = 0$, adding a constant gives you another possible antiderivative.
(constant)

$$\frac{d}{dx}(-\cos(x)) = -(-\sin(x)) = \sin(x)$$

so $\int \sin(x)\,dx = -\cos(x) + C$

$\int \tan(x)\,dx = ?$ We'll do a little later, using the Substitution Rule.

$\int \sec(x)\,dx = ?$ —— " —— & serious algebraic trickery.

3. $\int e^x\,dx = e^x + C$ since $\frac{d}{dx} e^x = e^x$

$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$ since $\frac{d}{dx}\left(\frac{a^x}{\ln(a)}\right) = \frac{\cancel{\ln(a)}\, a^x}{\cancel{\ln(a)}}$

$(a > 0)$

There also certain properties of (definite) integrals that we'll use:

(1) $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$

(c a constant)

and $\int c f(x)\,dx = c \int f(x)\,dx$

(2) $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

and $\int (f(x) + g(x))\,dx = \int f(x)\,dx \int g(x)\,dx$.

eg $\int_{-2}^{2} (4 - x^2)\,dx = \int_{-2}^{2} 4\,dx + \int_{-2}^{2} (-1) x^2\,dx = 4 \int_{-2}^{2} 1\,dx + (-1) \int_{-2}^{2} x^2\,dx$

$= 4 \cdot x \Big|_{-2}^{2} - \frac{x^{3}}{3} \Big|_{-2}^{2}$ $\xleftarrow{2+1}$

$= \cancel{\left[4 \cdot 2 - 4(-2)\right]}$

$= \left[\frac{2^3}{3} - \frac{(-2)^3}{3}\right]$

$[8 + 8]$

$= - \left[\frac{8}{3} + \frac{8}{3}\right]$

$= \frac{32}{3}$

Another common property of definite integrals:

(3) $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Also,

(4) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

& (5) $$\int_a^a f(x)\,dx = 0 .$$

Substitution Rule (basic form)

This is the reverse of the Chain Rule for derivatives.

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Alternatively, if $u = g(x)$ and $y = f(u)$,

then $$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{f'(u)} \cdot \frac{du}{dx}$$

$$\left[= f'(\underbrace{g(x)}_{u}) \cdot g'(x) \right].$$

In reverse this gives

$$\int_a^b h(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} h(u)\, du$$

and $u = g(x)$

so $du = g'(x)dx$

&

x	u
a	$g(a)$
b	$g(b)$

<u>Example:</u> $$\int_0^{\sqrt{\pi/2}} \underbrace{2x \cos(x^2)\, dx}_{du}$$

Note that $\frac{d}{dx}x^2 = 2x$,

so we try

$u = x^2 \Rightarrow du = 2x\,dx$

&

x	$u = x^2$
0	0
$\sqrt{\frac{\pi}{2}}$	$\frac{\pi}{2}$

$$= \int_0^{\pi/2} \cos(u)\, du$$

$$= \sin(u) \Big|_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0)$$

$$= 1 - 0 = 1 \checkmark$$