Taylor Series (§11.10) ①

or, writing functions as power series

We'll reverse engineer the formula for writing $f(x)$ as a power series $\sum_{n=0}^{\infty} a_n x^n$.

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

Plug in $x=0$: $f(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + \cdots = a_0$

So $a_0 = f(0)$.

Now observe that

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n x^n) = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$= 0 + a_1 + a_2 \cdot 2x + a_3 3x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Plug in $x=0$: $f'(0) = a_1$, i.e. $a_1 = f'(0)$ $\quad 1! = 1$

Do it again! $f''(x) = \frac{d}{dx} f'(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$= 2a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 a_4 x^2 + \cdots$$

Plug in $x=0$: $f''(0) = 2a_2$, i.e. $a_2 = \frac{f''(0)}{2}$ $\quad 2! = 2$

And one more time...

$$f'''(x) = f^{(3)}(x) = \frac{d}{dx} f''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3}$$

$$= 3\cdot 2\cdot 1 a_3 + 4\cdot 3\cdot 2 a_4 x + 5\cdot 4\cdot 3 a_5 x^2 + \cdots$$

Plug in $x=0$: $f^{(3)}(0) = 3!\, a_3$

ie $a_3 = \dfrac{f^{(3)}(0)}{3!}$

In general if $f^{(n)}(x)$ denotes the n^{th} derivative of $f(x)$, then

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

This also works for $n=0$, if we define $f^{(0)}(x) = f(x)$ & $0! = 1$.

So, <u>if</u> you can write $f(x)$ as a power series, that power series is

$$f(x) \sim \boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

<u>Taylor's formula</u>

This gives the power series equal to $f(x)$ [if there is one to be had].

Example: $f(x) = \cos(x)$ To find it's Taylor series we have to work out

$$a_n = \frac{f^{(n)}(0)}{n!}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos(x)$	$\cos(0) = 1$
1	$-\sin(x)$	$-\sin(0) = 0$
2	$-\cos(x)$	$-\cos(0) = -1$
3	$\sin(x)$	$\sin(0) = 0$
4	$\cos(x)$	$\cos(0) = 1$
5	$-\sin(x)$	$-\sin(0) = 0$
6	$-\cos(x)$	$-\cos(0) = -1$
7	$\sin(x)$	$-\sin(0) = 0$
8	$\cos(x)$	$\cos(0) = 1$
$\vdots$		$\vdots$

So $f^{(n)}(0) = 0$ when n is odd and alternates 1 & -1 when n is even.

ie if $n = 2k$,
then $f^{(2k)}(0) = (-1)^k$

Thus

$$a_n = \begin{cases} \frac{(-1)^k}{(2k)!} & \text{if } n = 2k \\ 0 & \text{otherwise} \end{cases}$$

So we'll write the Taylor series of $\cos(x)$ as

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

When does this Taylor series converge?

Ratio Test: $\lim_{k\to\infty}\left|\dfrac{\dfrac{(-1)^{k+1}x^{2(k+1)}}{(2(k+1))!}}{\dfrac{(-1)^k}{(2k)!}\cdot x^{2k}}\right|$ $\quad x^{2k}\geq 0$

$$= \lim_{k\to\infty}\left|\frac{(-1)^{\cancel{k+1}}x^{2k+2}}{(2k+2)!}\cdot\frac{\cancel{(2k)!}}{\cancel{(-1)^k}\cancel{x^{2k}}}\right|$$

$(2k+2)! = (2k+2)(2k+1)\cdot\cancel{(2k)!}$

$$= \lim_{k\to\infty}\frac{\cancel{1}\cdot 1\,x^2}{(2k+2)(2k+1)} \quad \begin{matrix}\to x^2\\ \to\infty\end{matrix} \quad = 0 < 1$$

so the series converges absolutely for all x

ie radius of convergence $r=\infty$
& interval of convergence $\cancel{i}$ $(-\infty,\infty)$

Example: $f(x) = \dfrac{1}{2+2x}$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\frac{1}{2+2x}$	$\frac{1}{2}\cdot 0!$
1	$\frac{-2}{(2+2x)^2}$	$-\frac{2}{2^2} = -\frac{1}{2}\cdot 1!$
2	$\frac{2^3}{(2+2x)^3}$	$\frac{2^3}{2^3} = 1 = \frac{+1}{2}\cdot 2!$
3	$\frac{-2^3\cdot 3\cdot 2}{(2+2x)^4}$	$\frac{-2^4 3}{2^4} = -3 = -\frac{1}{2}\cdot 3!$

$f'(x) = \frac{d}{dx}(2+2x)^{-1}$
$= (-1)(2+2x)^{-2}\frac{d}{dx}(2+2x)$
$= (-1)\cdot 2\,(2+2x)^{-2}$

$f''(x) = \frac{d}{dx}(-2)(2+2x)^{-2}$
$= (-2)(-2)(2+2x)^{-3}\,2$
$= 2^3(2+2x)^{-3}$

$f'''(x) = \frac{d}{dx}2^3(2+2x)^{-3}$
$= 2^3(-3)(2+2x)^{-4}\cdot 2$
$= -2^4 3\,(2+2x)^{-4}$

4 $\quad f^{(4)}(x) = \dfrac{2^4 \cdot 4!}{(2+2x)^5}$ $\quad f^{(4)}(0) = \dfrac{2^4 \cdot 4!}{2^5} = \frac{1}{2} \cdot 4!$

$$\frac{d}{dx}\left(-2^3 \cdot 3 \cdot 2 (2+2x)^{-4}\right) = -2^3(-4)3\cdot 2(2+2x)^{-5}\cdot 2 = 2^4 \cdot 4! \cdot (2+2x)^{-5}$$

$\vdots$

n $$f^{(n)}(0) = \frac{(-1)^n n!}{2}$$

So the series is $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!} / 2}{\cancel{n!}} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n x^n$$

(Note: $\dfrac{1}{2+2x} = \dfrac{1}{2} \cdot \dfrac{1}{1+x} = \dfrac{1}{2} \cdot \dfrac{1}{1-(-x)}$

$= \frac{1}{2}\left(1 - x + x^2 - x^3 + \cdots\right)$)

This converges for <u>some</u> x ?

Ratio Test: $\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n}}(-1)\,\cancel{x^{n}}x}{\cancel{(-1)^n}\cancel{x^n}}\right|$

$= \lim_{n\to\infty}|(-1)x| = |x|$

So the series converges absolutely when $|x|<1$ & diverges when $|x|>1$. At $|x|=1$, ie $x=\pm 1$, the Ratio Test is inconclusive.

At $x=1$, $\sum_{n=0}^{\infty}(-1)^n 1^n = \sum_{n=0}^{\infty}(-1)^n = 1-1+1-1+\cdots$

which diverges by the Divergence Test since $\lim_{n\to\infty}|(-1)^n| = \lim_{n\to\infty} 1 = 1 \neq 0$

At $x=-1$, $\sum_{n=0}^{\infty}(-1)^n(-1)^n = \sum_{n=0}^{\infty} 1$

which diverges by the Divergence Test since $\lim_{n\to\infty} 1 = 1 \neq 0$.

The radius of convergence of the Taylor series $\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n$ of $f(x) = \frac{1}{2+2x}$ is $r=1$ & the interval of convergence is $(-1,1)$.

$f(x)$ is defined for all x except $x=-1$.

Something else that can go wrong:

The Taylor series could be defined and converge for all x but fail to be equal to the function except at 0.

Example: (Cauchy)

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Fact: $f^{(n)}(0)$ is defined for all $n \geq 0$ and $f^{(n)}(0) = 0$.

So the Taylor series of $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = \sum_{n=0}^{\infty} 0$$

This converges for all x to the function $g(x) = 0$.

... which is different from $f(x)$ except at $x = 0$.

Most functions you will ever see do have the Taylor series converge to the function it came from...

Dirty tricks with Taylor series

1. If you can find a powerseries that is equal to $f(x)$, it is the Taylor series of $f(x)$

es $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$ sum of a geometric series with first term 1 & common ratio $-x^2$

$= 1 - x^2 + x^4 - x^6 + \cdots$

so this is the Taylor series of $f(x) = \frac{1}{1+x^2}$.

2. If you know the Taylor series of $f(x)$,

then 1) the Taylor series of $f'(x)$ is the derivative of the Taylor series of $f(x)$

& 2) the Taylor series of $\int f(x)\,dx$ is the integral (up to a constant) of the Taylor series of $f(x)$

You can expand functions as Taylor series at some point $a \neq 0$ as well

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Usually save this for functions not defined at 0, eg $\ln(x)$.

This is the end!

For those interested check out §11.11 for how to use partial sums of Taylor series to approximate functions and estimate the error.