Power Series (§11.8 & 11.9)

or, "polynomials" of infinite degree

A power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where each a_n is a constant as far as x is concerned. (a_n may depend on n, but not on x.)

Prototype: The geometric series with first term $a \in \mathbb{R}$ and common ratio $r = x$.

$$a + ax + ax^2 + ax^3 + \cdots = \sum_{n=0}^{\infty} a x^n = a\left(\sum_{n=0}^{\infty} x^n\right)$$

As long as $|r| = |x| < 1$, this converges (absolutely) and it diverges when $|r| = |x| \geq 1$. When it converges, it does so to $\frac{a}{1-x}$.

Example: Exponential series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$= e^x$ when the series converges

Q: When does it converge?

First use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}}{\underbrace{(n+1)!}_{(n+1)\cdot\cancel{n!}}}\cdot\frac{\cancel{n!}}{\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = \lim_{n\to\infty}\frac{|x| \to |x|}{n+1 \to \infty} = 0 < 1$$

so, by the Ratio Test, this converges absolutely no what value x has. i.e. "It converges absolutely for all x."

So, for all x, $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$.

Example: $\sum_{n=0}^{\infty}\frac{(n+1)^2}{2^{n+1}}x^n$

Ratio Test: $$\lim_{n\to\infty}\left|\frac{\frac{(n+2)^2x^{n+1}}{2^{n+2}}}{\frac{(n+1)^2x^n}{2^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)^2x^{n+1}}{2^{n+2}}\cdot\frac{2^{n+1}}{(n+1)^2x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{2}\cdot\frac{n^2+4n+4}{n^2+2n+1}\right| = \frac{|x|}{2}\lim_{n\to\infty}\frac{n^2+4n+4}{n^2+2n+1}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \frac{|x|}{2}\lim_{n\to\infty}\frac{1+\frac{4}{n}+\frac{4}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} = \frac{|x|}{2}\cdot\frac{1+0+0}{1+0+0} = \frac{|x|}{2}$$

By the Ratio Test the series converges absolutely when $\frac{|x|}{2} < 1$ & diverges when $\frac{|x|}{2} > 1$.

So we have convergence when x is in $(-2,2)$

$$\frac{|x|}{2} < 1$$
$$\iff |x| < 2$$
$$\iff -2 < x < 2$$

& divergence when x is in $(-\infty,-2) \cup (2,\infty)$

ie when $\frac{|x|}{2} > 1$

ie $|x| > 2$

The Ratio Test tells us nothing when $\frac{|x|}{2} = 1$ ie $|x| = 2$ ie $x = \pm 2$, so we have to test these cases separately

$x=-2$: The series is
$$\sum_{n=0}^{\infty} \frac{(n+1)^2}{2^{n+1}}(-2)^n$$
$$= \sum_{n=0}^{\infty} \frac{(n+1)^2}{2^{\cancel{n+1}}}(-1)^n \cancel{2^n}$$
$$= \sum_{n=0}^{\infty} \frac{(n+1)^2(-1)^n}{2} = \frac{1}{2} - 2 + \frac{9}{2} - \cdots$$

This diverges by the Divergence Test because
$$\lim_{n\to\infty} \left| \frac{(n+1)^2(-1)^n}{2} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{2} \begin{matrix} \to \infty \\ \to 2 \end{matrix} = \infty \neq 0.$$

$x=2$: The series is $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^2 \cancel{2^n}}{2^{\cancel{n}+1}} = \sum_{n=0}^{\infty} \frac{(n+1)^2}{2}$.

This also diverges by the Divergence Test:

$$\lim_{n\to\infty} \left| \frac{(n+1)^2}{2} \right| = \lim_{n\to\infty} \frac{(n+1)^2 \to \infty}{2 \to 2} = \infty \neq 0$$

In general a "Power Series" converges on some interval $(-r,r)$ and diverges outside $[-r,r]$ (assuming $r<\infty$) and may converge or diverge (separately!) at the endpoints $x=\pm r$ [assuming $(-r,r)$.

r = <u>radius of convergense</u>

and the actual interval (including any convergent endpoints) is the <u>interval of convergence</u>.

<u>eg</u>

$\displaystyle\sum_{n=0}^{\infty} x^n$ $\quad r=1$ & interval of convergence $(-1,1)$

$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad r=\infty$ & —— " —————— $(-\infty,\infty)$

$\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^2}{2^{n+1}} x^n$ $\quad r=2$ & interval of convergence $(-2,2)$

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ this series has $r=1$

& interval of convergence is $[-1,1)$

(because at $x=-1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which conditionally converges, and at $x=1$ ~~the~~ ~~t~~ we have $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges)

Within the radius of convergence we can safely differentiate and integrate a power series term-by-term.

eg $\quad 1+x+x^2+\cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $|x|<1$.

Inside the radius of convergence is $|x|<1$

$$\frac{d}{dx}\left(1+x+x^2+\cdots\right) = 0+1+2x+3x^2+\cdots$$

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}\frac{d}{dx}(1-x)$$

$$= \cancel{(-1)}(1-x)^{-2}\cancel{(-1)}$$

$$= (1-x)^{-2} = \frac{1}{(1-x)^2}$$

so $\sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

$= 0+1+2x+3x^2+\cdots$