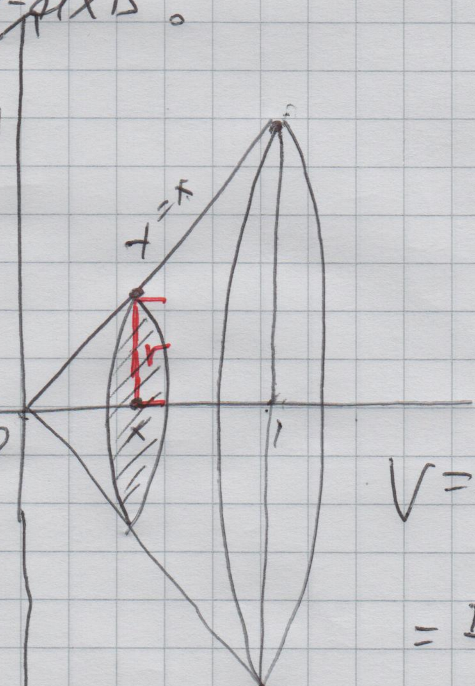
Applications II

(Covering part of § 9.3 & § 9.10.)

[volumes of solids of revolution and
areas of surfaces of revolution]

A solid of revolution is obtained by revolving a region about a line. (All the way around the line.

es Revolve the region bordered by $y=x$ for $0\le x\le 1$, the x-axis, for $0\le x\le 1$, and the line $x=1$, for $0\le y\le 1$, about the x-axis.



Cone

with "base" radius 1 & height 1

$V=\frac{1}{3}\pi r^2 h$

This is an example of the <u>disk method</u>.

Use cross-sections

$A(x)=\pi r^2$

$=\pi(y-0)^2$

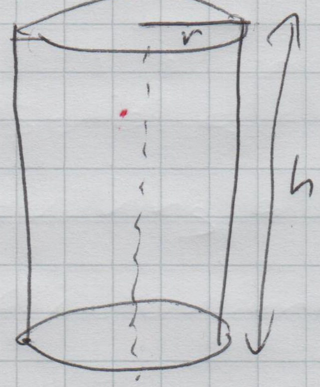
$=\pi x^2$

$V=\int_0^1 A(x)\,dx=\int_0^1 \pi x^2\,dx$

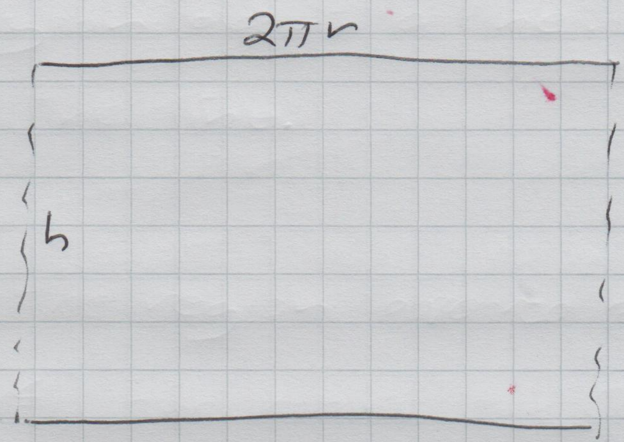
$=\frac{\pi x^3}{3}\Big|_0^1=\frac{\pi\cdot 1^3}{3}-\frac{\pi\cdot \cancel{0}^3}{\cancel{3}}$

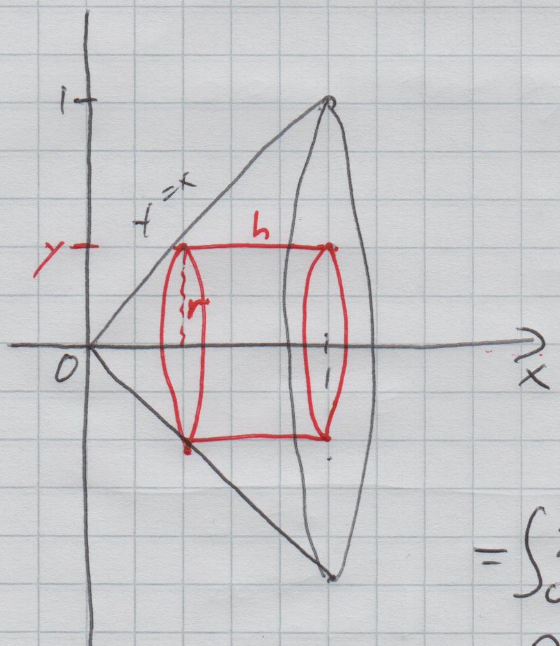
$=\frac{\pi}{3}$

We could have done the same problem using cylindrical "cross-sections".



Area of a cylinder (without the ends) is $2\pi rh$





Always use the variable that runs perpendicular to your cross-sections.

$0 \le y \le 1$ over the region

$$V = \int_0^1 2\pi r h \, dy$$

$r = y - 0 = y$

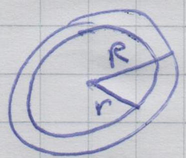
$h = 1 - x$
$\quad = 1 - y$

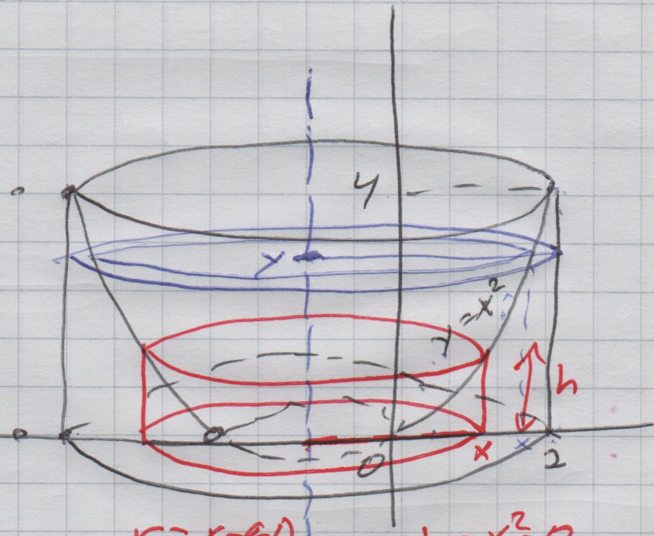
$$= \int_0^1 2\pi y(1-y)\,dy$$

$$= 2\pi \int_0^1 (y - y^2)\,dy = 2\pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1$$

$$= 2\pi\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \cancel{2\pi\left(\frac{0^2}{2} - \frac{0^3}{3}\right)}$$

$$= 2\pi\left(\frac{1}{2} - \frac{1}{3}\right) = 2\pi \cdot \frac{1}{6} = \frac{\pi}{3} \checkmark$$





Region below $y = x^2$ & above $y = 0$ for $0 \le x \le 2$

Revolve it about $x = -1$.

$R = 2-(-1) = 2+1 = 3$

$r = x-(-1) = x+1 = \sqrt{y}+1$

$r = x-(-1) = x+1$ $\quad$ $h = x^2 - 0 = x^2$

$x = -1$

Using cylindrical shells

$$V = \int_0^2 2\pi r h\,dx = 2\pi\int_0^2 (x+1)x^2\,dx = 2\pi\int_0^2 (x^3+x^2)\,dx$$

$$= 2\pi\left(\frac{x^4}{4}+\frac{x^3}{3}\right)\Big|_0^2 = 2\pi\left(\frac{2^4}{4}+\frac{2^3}{3}\right) - \cancel{2\pi\left(\frac{0^4}{4}+\frac{0^3}{3}\right)}$$

$$= 2\pi\left(\frac{16}{4}+\frac{8}{3}\right) = 2\pi\left(4+\frac{8}{3}\right) = 2\pi\left(\frac{12}{3}+\frac{8}{3}\right)$$

$$= 2\pi\cdot\frac{20}{3} = \frac{40}{3}\pi$$

We could use disks/washers?

Washer has area $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

$= \pi(3^2 - (\sqrt{y}+1)^2)$

$0 \le y \le 4$

$$V = \int_0^4 \pi(R^2 - r^2)\,dy = \pi\int_0^4 (9-(y+2\sqrt{y}+1))\,dy$$

$$= \pi\int_0^4 (-y-2y^{1/2}+8)\,dy = \pi\left(-\frac{y^2}{2} - 2\frac{y^{3/2}}{3/2} + 8y\right)\Big|_0^4$$

$$= \pi\left(-\frac{y^2}{2} - \frac{4}{3}y^{3/2} + 8y\right)\Big|_0^4 = \pi\left(-\frac{16}{2} - \frac{4}{3}\cdot 8 + 8\cdot 4\right) - 0$$

$$= \pi\left(-8 - \frac{32}{3} + 32\right) = \pi\left(24 - \frac{32}{3}\right) = \pi\left(\frac{72-32}{3}\right) = \frac{40}{3}\pi$$

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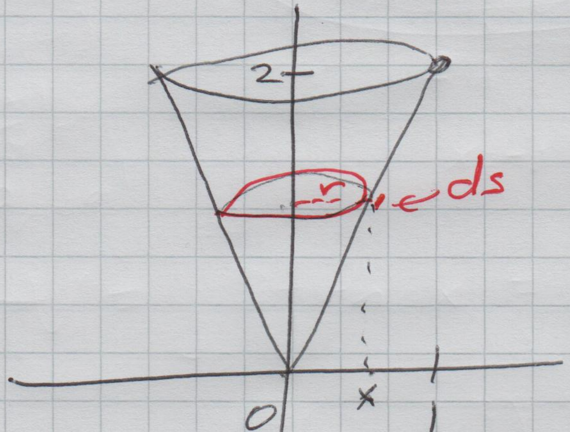
Surface areas

(areas of surfaces of revolution)

Revolve a curve about a line. (All the way around.)

eg Surface area of cone (not counting the base)



Revolve $y=2x$, $0\le x\le 1$, about the y-axis.

cylinder of height ds & radius r contributes $2\pi r\,ds$ — ds: piece of arc-length

$r = x - 0 = x$

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$\frac{dy}{dx} = \frac{d}{dx}(2x) = 2$

$$SA = \int_0^1 2\pi r\,ds = 2\pi\int_0^1 x\sqrt{1+2^2}\,dx$$

$$= 2\sqrt{5}\cdot\pi\int_0^1 x\,dx = \cancel{2}\sqrt{5}\pi\cdot\frac{x^2}{\cancel{2}}\Big|_0^1 = \sqrt{5}\pi(1^2-0^2)$$

$$= \sqrt{5}\cdot\pi$$

In terms of y, $0\le y\le 2$, $x=\frac{y}{2}$ so $\frac{dx}{dy}=\frac{1}{2}$

$r = x = \frac{y}{2}$ $\quad ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{1+\frac{1}{4}}\,dy = \frac{\sqrt{5}}{2}\,dy$

$$SA = \int_0^2 2\pi r\,ds = \cancel{2}\pi\int_0^2 \frac{y}{\cancel{2}}\cdot\frac{\sqrt{5}}{2}\,dy = \frac{\sqrt{5}}{2}\pi\int_0^2 y\,dy$$

$$= \frac{\sqrt{5}}{2}\pi\,\frac{y^2}{2}\Big|_0^2 = \frac{\sqrt{5}}{\cancel{2}}\pi\cdot\frac{2^{\cancel{2}}}{\cancel{2}} - \cancel{\frac{\sqrt{5}}{2}\pi\cdot\frac{0^2}{2}}$$

$$= \sqrt{5}\pi$$