Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2020

Solutions to Assignment #3
Exponential and Differential

Just in case you haven't seen it before, or have forgotten about it, the notation $n!$ is a shorthand for the product of the first n positive integers, that is:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Thus $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and so on. $n!$ grows very quickly, faster than any exponential function with a constant base. (Stirling's Formula tells us that when n is large, $n!$ is approximately $\sqrt{2n\pi} \cdot \dfrac{n^n}{e^n}$.)

This notation is extended to $n = 0$ by defining $0! = 1$. This is mainly done to make various general formulas and expressions involving $n!$ (including the sum in question **2** below) behave nicely when $n = 0$. One could also justify this by observing that $n!$ counts the number of ways one can arrange n distinct objects in a row, and that there is only one way of arranging no objects at all ...

1. Suppose $y = f(x)$ satisfies the equation $\dfrac{dy}{dx} = y$. Show that $f(x) = Ke^x$ for some constant K. *[5]*

SOLUTION. First, note that $y = f(x) = 0$ for all x is a solution to the given differential equation because $\frac{d}{dx} 0 = 0$. Then $f(x) = Ke^x = 0e^x = 0$ for $K = 0$.

Now suppose that $y = f(x)$ is differentiable and not equal to 0 for some value(s) of x. At least for such values, we can then rearrange the differential equation as follows,

$$\frac{dy}{dx} = y \implies \frac{1}{y} \cdot \frac{dy}{dx} = 1,$$

and then compute the antiderivative of both sides. The right-hand side is easy: $\int 1\, dx = x + C$ by the Power Rule.

For the left-hand side, a quick and dirty approach would be to do the following:

$$\int \frac{1}{y} \cdot \frac{dy}{dx}\, dx = \int \frac{1}{y}\, dy = \ln(y) + B$$

(We use B because we've already used C for the generic constant of integration on the right-hand side.) This is one of those cases where one gets away with treating $\dfrac{dy}{dx}$ as if it were really a fraction.

A nominally more careful (and mathematically respectable :-) approach would be to treat this as an opportunity for a trivial subsitution $u = y$, so $du = \dfrac{dy}{dx}\, dx$:

$$\int \frac{1}{y} \cdot \frac{dy}{dx}\, dx = \int \frac{1}{u}\, du = \ln(u) + B = \ln(y) + B$$

Respectably or otherwise, we have arrived at $\ln(y)+B = x+C$. Solving this equation for y yields:

$$\ln(y) + B = x + C \implies \ln(y) = x + C - B \implies y = e^{\ln(y)} = e^{x+C-B} = e^{C-B}e^{x}$$

Setting $K = e^{C-B}$ means that $y = f(x) = Ke^x$ has the desired form. Note that making $K = -e^{C-B}$ works too, since the negative sign will pass through the derivative and hence appear on both sides of the differential equation. (Alternatively, one could exploit the fact that $\ln(|y|)$ is a more general antiderivative of $\frac{1}{y}$ and eventually get $K = \pm e^{C-B}$.)

Thus, whether or not $y = f(x)$ is always 0, if it is a solution of the differential equation $\frac{dy}{dx} = y$, we must have $y = f(x) = Ke^x$ for some constant K. $\square$

2. Suppose $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$. Use **1** (and just a bit more) to show that $f(x) = e^x$. *[5]*

NOTE. For the sake of this assignment, you may assume that the sum $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ makes sense no matter what the value of x is. We'll see exactly what this means and how to check it is so later in the course. For now, just think of the sum as a polynomial of infinite degree.

SOLUTION. One thing we can do with polynomials is differentiate them term-by-term. Following the hint and thinking of the series as a polynomial of infinite degree, we differentiate it term-by=term too:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \\
&= \frac{d}{dx}\left(\frac{x^0}{0!}\right) + \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) \\
&= \frac{d}{dx}1 + \frac{d}{dx}x + \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}\left(\frac{x^3}{6}\right) + \frac{d}{dx}\left(\frac{x^4}{24}\right) + \cdots \\
&= 0 + \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x)
\end{aligned}$$

Thus $y = f(x)$ is equal to its derivative, *i.e.* $\frac{dy}{dx} = y$, so **1** tells us that $f(x) = Ke^x$ for some constant K. Since $f(0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} = 1 + 0 + \frac{0^2}{2} + \frac{0^3}{6} + \cdots = 1 + 0 = 1$ (however many 0s you add, you're not going to get much :-), it follows that $K = K \cdot 1 = Ke^0 = f(0) = 1$. Thus $f(x) = e^x$, *i.e.* $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$, as desired. $\blacksquare$