Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Summer 2020

Solutions to Quiz #5
Tuesday, 21 July.

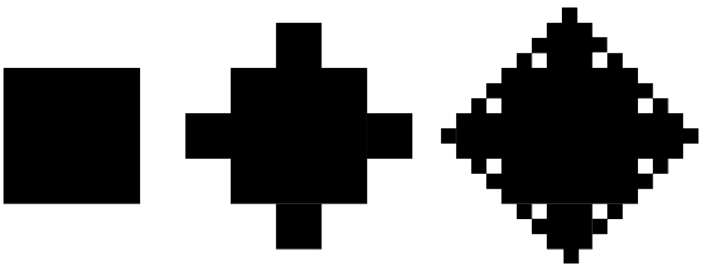
We will construct a two-dimensional shape as follows:

At stage 0 we take a unit square, *i.e.* one with sides of length 1.

At stage 1, we attach a square with sides of length $\frac{1}{3}$ to the middle of each side of the unit square.

At stage 2, we attach a square with sides of length $\frac{1}{9}$ to the middle of each side of the shape obtained in stage 1.

Here are the shapes we have at stages 0, 1, and 2, respectively:



In general, at stage $n+1$, we attach a square with sides $\left(\frac{1}{3}\right)^{n+1}$ to the middle of each side of the shape obtained at stage n.

Consider the object that you would have at the end of this (infinite!) process.

1. What should the length of the border of the limit object be? *[2]*

Solution. How does the border change from one step to the next? We replace each line segment of the border by five line segments, each of which is one third the length of the replaced one. It follows that the length of the border of the shape at stage $n+1$ is $\frac{5}{3}$ of the length of the border of the shape at stage n. Since the unit square we start with at stage $n=0$ has a border that is $4 \cdot 1 = 4$ units long, the length of the border of the shape at stage n is going to be $4\left(\frac{5}{3}\right)^n$ units. The length of the border of the limit shape should then be $\displaystyle\lim_{n\to\infty} 4\left(\frac{5}{3}\right)^n = \infty$, as $\left(\frac{5}{3}\right)^n \to \infty$ as $n \to \infty$ because $\frac{5}{3} > 1$. $\square$

2. What should the area of the limit object be? *[2.5]*

Solution. We first work out how many line segments make up the border of the shape at stage n. As noted in the solution to **1** above, to go from one stage to the next, we replace each line segment by five smaller ones, each of one third the length of the replaced one. Since the square at stage 0 has four line segments of length 1 making up its border, it is not hard to see that at stage n we have $4 \cdot 5^n$ line segments of length $\left(\frac{1}{3}\right)^n$ making up the border. At stage $n+1$ we then add to the area of the shape at stage n by attaching $4 \cdot 5^n$ little squares of side length $\left(\frac{1}{3}\right)^{n+1}$, and hence adding area amounting to

$4 \cdot 5^n \cdot \left[\left(\frac{1}{3}\right)^{n+1}\right]^2 = 4 \cdot 5^n \cdot \left(\frac{1}{3}\right)^{2n+2} = 4 \cdot 5^n \cdot \left(\frac{1}{9}\right)^{n+1} = \frac{4}{9} \cdot \left(\frac{5}{9}\right)^n$. Thus the shape at stage $n+1$ has a total area of

$$1 + \frac{4}{9} \cdot \left(\frac{5}{9}\right)^0 + \frac{4}{9} \cdot \left(\frac{5}{9}\right)^1 + \frac{4}{9} \cdot \left(\frac{5}{9}\right)^2 + \cdots + \frac{4}{9} \cdot \left(\frac{5}{9}\right)^n = 1 + \sum_{k=0}^{n} \frac{4}{9} \cdot \left(\frac{5}{9}\right)^k ,$$

where the 1 at the beginning is the area of the unit square at stage 0. As we take the limit as $n \to \infty$ the area of the limit shape is given by the infinite series

$$1 + \sum_{k=0}^{\infty} \frac{4}{9} \cdot \left(\frac{5}{9}\right)^k = 1 + \frac{4}{9} \cdot \left(\frac{5}{9}\right)^0 + \frac{4}{9} \cdot \left(\frac{5}{9}\right)^1 + \frac{4}{9} \cdot \left(\frac{5}{9}\right)^2 + \cdots .$$

After the 1 this is a geometric series with first term $\frac{4}{9}$ and common ration $r = \frac{5}{9}$. Since $|r| = \frac{5}{9} < 1$, this geometric series converges, and the sum formula $\frac{a}{1-r}$ tells us that the area of the limit shape is therefore:

$$1 + \frac{\frac{4}{9}}{1 - \frac{5}{9}} = 1 + \frac{\frac{4}{9}}{\frac{4}{9}} = 1 + 1 = 2 \qquad \square$$

Note: There are a lot of other shapes that have finite area but an infinitely long border. The *Koch curve*, often called the *snowflake curve* or *Koch snowflake*, is a particularly nice-looking one, and is obtained by a process similar to the one used above.

3. What shape is the limit object very close to being? *[0.5]*

Solution. It's basically a square with sides of length $\sqrt{2}$, with a pattern of infinitely many points missing. (Hence the length of the border being very large …). ■